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ON THE K-THEORY OF LAURENT POLYNOMIALS

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ABSTRACT. The Karoubi-Villamayor K-theory of the ring of Laurent polynomials over a regular ring is computed. It is shown that Milnor's K_2 of a ring of Laurent polynomials over a regular ring maps onto K_1 of the ring.

In [3] we introduced an algebraic K-theory, $\{\kappa_i^{\mathrm{GL}}\}$, and discussed its general properties. In this article we compute $\kappa_i^{\mathrm{GL}}(R[x, x^{-1}])$ for regular rings. More precisely we establish

THEOREM. *If R is left regular, then* $\kappa_{n+1}^{\mathrm{GL}}(R[x, x^{-1}]) \cong \kappa_{n+1}^{\mathrm{GL}}(R) \oplus \kappa_n^{\mathrm{GL}}(R)$.

This result should be viewed as a corollary of the techniques of [2] interpreted in the light of the formalism of [3].

As a corollary, we prove that if we write $K_2(R[x, x^{-1}]) = K_2(R) \oplus X$, then if R is left regular, there is a surjection $X \twoheadrightarrow K_1(R)$. H. Bass has informed me that J. Wagoner also has results about the group X above.

We shall follow the notations of [3] throughout this article. The results will be used in a subsequent paper to study in greater detail the relationship between K_2 and κ_2^{GL}.

1. Polynomial extensions.

THEOREM 1.1. *If R is any ring (without unit) and $R[x]$ is the polynomial ring in one variable x, then the inclusion $R \to R[x]$ induces an isomorphism*

$$\kappa_i^{\mathrm{GL}}(R) \cong \kappa_i^{\mathrm{GL}}(R[x])$$

for all $i \geqq 1$.

PROOF. The result is given in [3, §3, Remark 1]. Here is the argument. One considers the s.e.s.

$$ER \to R[x] \to R,$$

a GL fibration since it splits. The long exact sequence [3, Proposition 2.12 and 5.1], yields

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$$\kappa_i^{\mathrm{GL}}(R[x]) \cong \kappa_i^{\mathrm{GL}}(R) \oplus \kappa_i^{\mathrm{GL}}(ER).$$

Also, by [**3**, 3.4 and 5.2], $\kappa_i^{\mathrm{GL}}(ER)=0$ $(i\geq 1)$ from which follows the theorem.

Observe that at the K_0 level we have

$$\overline{K}_0(R[x]^+) = \overline{K}_0(R^+) \oplus \overline{K}_0(ER^+).$$

If R is left regular (i.e., unital, left noetherian, and each *fg* module has finite h.d.), then Grothendieck's theorem [**2**] implies that $\overline{K}_0(ER^+)=0$.

For ease of notation, denote $\overline{K}_0(A^+)$ by $K_0(A)$ for any ring A. This notation is consistent for if A has a unit, $A^+ \cong A\pi\mathbf{Z}$ and $K_0(A^+)=K_0(A)\oplus K_0(\mathbf{Z})$.

Proposition 1.2. *Let R be a ring and suppose that $R\to R[X]$ induces an isomorphism $K_0(R)\to K_0(R[X])$ for any set X, where $R[X]$ is the polynomial ring. Then $K_0(ER[X])=0$.*

Proposition 1.3. *Suppose that $K_0(R)\to K_0(R[X])$ is an isomorphism for all X. Then $K_0(\Omega R)\to K_0(\Omega R[X])$ is an isomorphism.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \Omega R & \longrightarrow & ER & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ \Omega R[X] & \rightarrow & ER[X] & \rightarrow & R[X] \end{array}$$

whose rows are s.e.s.'s. By [**3**, 3.5], $ER\to R$ and $ER[X]\to R[X]$ are GL-fibrations ($(ER)[X]=E(R[X])$). Hence by [**3**, 5.10, 5.8], we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \kappa_1^{\mathrm{GL}}(ER) & \longrightarrow & \kappa_1^{\mathrm{GL}}(R) & \longrightarrow & K_0(\Omega R) & \longrightarrow & K_0(ER)\rightarrow \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \kappa_1^{\mathrm{GL}}(ER[X]) & \rightarrow & \kappa_1^{\mathrm{GL}}(R[X]) & \rightarrow & K_0(\Omega R[X]) & \rightarrow & K_0(ER[X]) \end{array}$$

The two left vertical arrows are isomorphisms by 1.1 and the fact that κ_1^{GL} commutes with direct limits. The result follows by 1.2 and the five lemma.

Corollary 1.4. If $K_0(R)\to K_0(R[X])$ *is an isomorphism for all X, then $K_0(E\Omega^n R)=0$ for all n. In particular this is the case if R is left regular.*

Corollary 1.5. *Under the hypotheses of* 1.4, $\kappa_n^{\mathrm{GL}}(R)\cong K_0(\Omega^n R)$.

PROOF. We deduce by "dimension shifting" that $\kappa_n^{\mathrm{GL}}(R) \cong \kappa_1^{\mathrm{GL}}(\Omega^{n-1}R)$. The isomorphism $\kappa_1^{\mathrm{GL}}(\Omega^{n-1}R) \cong K_0(\Omega^n R)$ follows from 1.4 and the long exact sequence associated to the s.e.s.

$$\Omega^n R \to E\Omega^{n-1}R \to \Omega^{n-1}R.$$

2. **Laurent polynomials** $R[x, x^{-1}]$. We pass now to a discussion of $\kappa_i^{\mathrm{GL}}(R[x, x^{-1}])$ where $R[x, x^{-1}]$ is the group ring of the infinite cyclic group generated by x. Let us recall the main results of [2].

If A is a unital ring, then there is a group homomorphism

$$\mathrm{GL}(A[x, x^{-1}]) \xrightarrow{\phi_A} K_0(A)$$

which induces a split epimorphism

$$K_1(A[x, x^{-1}]) \to K_0(A).$$

Furthermore ϕ_A is natural in A in the sense that if $f: A \to B$ is a unital ring homomorphism, then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{GL}(A[x, x^{-1}]) & \xrightarrow{\phi_A} & K_0(A) \\ \downarrow \mathrm{GL}(f[x, x^{-1}]) & & \downarrow K_0(f) \\ \mathrm{GL}(B[x, x^{-1}]) & \xrightarrow{\phi_B} & K_0(B) \end{array}$$

In addition, $K_1(A[x, x^{-1}]) \cong K_0(A) \oplus K_1(A) \oplus X_A$ where X_A is generated by unipotents and the decomposition is natural in A.

We indicate now the modifications necessary if R is a ring without unit. Recall that $\mathrm{GL}(R)$ is an invariant subgroup of $\mathrm{GL}(R^+)$. We define $\phi_R: \mathrm{GL}(R[x, x^{-1}]) \to K_0(R)$ to make the following diagram commute

$$\begin{array}{ccccc} \mathrm{GL}(R[x, x^{-1}]) & & \xrightarrow{\phi_R} & & K_0(R) \\ \downarrow \subseteq & & & & \uparrow \\ \mathrm{GL}(R[x, x^{-1}]^+) & \longrightarrow & \mathrm{GL}(R^+[x, x^{-1}]) & \xrightarrow{\phi_{R^+}} & K_0(R^+) \end{array} \tag{2.1}$$

Here, the map $K_0(R^+) \to K_0(R) = \overline{K}_0(R^+) = \mathrm{Ker}(K_0(R^+) \to K_0(\mathbf{Z}))$ is the obvious retraction. Evidently ϕ_R so defined is natural in R, and this notation is consistent if R has a unit.

We need to determine the precise relationship between $\kappa_1^{\mathrm{GL}}(R[x, x^{-1}]^+)$ and $\kappa_1^{\mathrm{GL}}(R^+[x, x^{-1}])$. Observe that the diagram

$$\begin{array}{ccc} R[x, x^{-1}]^{+} & \longrightarrow & \boldsymbol{Z} \\ \downarrow & & \downarrow \\ R^{+}[x, x^{-1}] & \longrightarrow & \boldsymbol{Z}[x, x^{-1}], \end{array}$$

whose horizontal arrows are augmentations and vertical arrows are inclusions, is cartesian, and the horizontal arrows are split. Then the Mayer-Vietoris sequence [**3**, §2, §5], becomes

$$0 \to \kappa_1^{\mathrm{GL}}(R[x, x^{-1}]^{+}) \to \kappa_1^{\mathrm{GL}}(\boldsymbol{Z}) \oplus \kappa_1^{\mathrm{GL}}(R^{+}[x, x^{-1}]) \to \kappa_1^{\mathrm{GL}}(\boldsymbol{Z}[x, x^{-1}]) \to 0.$$

Thus $\kappa_1^{\mathrm{GL}}(R[x, x^{-1}]) = \mathrm{Ker}(\kappa_1^{\mathrm{GL}}(R^{+}[x, x^{-1}]) \to \kappa_1^{\mathrm{GL}}(\boldsymbol{Z}[x, x^{-1}]))$. Since $K_1(R^{+}[x, x^{-1}]) = K_1(R^{+}) \oplus K_0(R^{+}) \oplus X_{R^{+}}$ and since $\kappa_1^{\mathrm{GL}}(\boldsymbol{Z}[x, x^{-1}]) = K_1(\boldsymbol{Z}) \oplus K_0(\boldsymbol{Z})$, if we knew that passage from $K_1(R^{+}[x, x^{-1}])$ to $\kappa_1^{\mathrm{GL}}(R^{+}[x, x^{-1}])$ did not kill any of the $K_0(R)$ factor, we could conclude that

$$\kappa_1^{\mathrm{GL}}(R^{+}[x, x^{-1}]) = \kappa_1^{\mathrm{GL}}(R^{+}) \oplus K_0(R^{+})$$

and hence

$$\kappa_1^{\mathrm{GL}}(R[x, x^{-1}]) = \kappa_1^{\mathrm{GL}}(R) \oplus K_0(R).$$

THEOREM 2.2. *Suppose that R is left regular (or more generally that $K_0(R) \to K_0(R[X])$ is an isomorphism for all sets X). Then for each $n \geqq 0$ the map*

$$\phi_{\Omega^n R}\colon \mathrm{GL}(\Omega^n R[x, x^{-1}]) \to K_0(\Omega^n R)$$

factors through $\kappa_1^{\mathrm{GL}}(\Omega^n R[x, x^{-1}])$. Furthermore, we have

$$\kappa_{n+1}^{\mathrm{GL}}(R[x, x^{-1}]) \cong \kappa_{n+1}^{\mathrm{GL}}(R) \oplus \kappa_n^{\mathrm{GL}}(R) \tag{2.3}$$

(if $n=0$, $\kappa_0^{\mathrm{GL}} = K_0$).

PROOF. Recall from [**3**, §5], that $\kappa_1^{\mathrm{GL}}(R) = \mathrm{GL}(R)/\mathrm{GL}(\partial)\mathrm{GL}(ER)$, $ER = tR[t]$, and $\partial\colon ER \to R$ is given by "$t \to 1$." Furthermore, $E(\Omega^n R[x, x^{-1}]) = (E\Omega^n R)[x, x^{-1}]$. Consider now the commutative diagram

$$\begin{array}{ccc} \mathrm{GL}(E\Omega^n R[x, x^{-1}]) & \xrightarrow{\phi_{E\Omega^n R}} & K_0(E\Omega^n R) \\ \downarrow \mathrm{GL}(\partial) & & \downarrow K_0(\partial) \\ \mathrm{GL}(\Omega^n R[x, x^{-1}]) & \xrightarrow{\phi_{\Omega^n R}} & K_0(\Omega^n R) \end{array}$$

By 1.4, we have $K_0(E\Omega^n R)=0$. It follows from the preceding description of κ_1^{GL} that $\phi_{\Omega^n R}$ factors through $\kappa_1^{\mathrm{GL}}(\Omega^n R[x, x^{-1}])$.

To establish (2.3), we apply (2.1) to $\Omega^n R$ and observe that passage from K_1 to κ_1^{GL} kills all unipotents, but does not disturb the K_0 factor, by the preceding paragraph. Thus

$$\kappa_1^{\mathrm{GL}}(\Omega^n R[x, x^{-1}]) \cong \kappa_1^{\mathrm{GL}}(\Omega^n R) \oplus K_0(\Omega^n R).$$

The proof is completed now by "dimension shifting" and by applying 1.5.

COROLLARY 2.4. *If R is left regular and if we write $K_2(R[x, x^{-1}]) = K_2(R) \oplus X$, then there is a surjection $X \to K_1(R)$.*

PROOF. The map $\psi: K_2(R[x, x^{-1}]) \to \kappa_2^{\mathrm{GL}}(R[x, x^{-1}])$ of [**3**, 6.1], is surjective since $R[x, x^{-1}]$ is left regular. Also $\kappa_2^{\mathrm{GL}}(R[x, x^{-1}]) \cong \kappa_2^{\mathrm{GL}}(R) \oplus \kappa_1^{\mathrm{GL}}(R)$ by 2.3. The result follows from the observation that $\kappa_1^{\mathrm{GL}}(R) = K_1(R)$ if R is left regular, a consequence of [**2**, Theorem 1].

3. **K_0 is not a homotopy functor.** This question arises since κ_i^{GL} is a homotopy functor for $i \geqq 1$ (see [**3**, §3]). We observe that

$$\partial_x^1 \simeq \partial_x^0 : R[x] \to R$$

where ∂_x^i is "evaluation at i," $i=0, 1$. Furthermore, $K_0(R[x]) = K_0(R) \oplus K_0(ER)$ where $K_0(ER) = \mathrm{Ker}\ K_0(\partial_x^0): K_0(R[x]) \to K_0(R)$. Thus to show that K_0 is not a homotopy functor we need only exhibit an element $\eta \in K_0(R[x])$ such that $K_0(\partial_x^0)\eta = 0$ but $K_0(\partial_x^1)\eta \neq 0$. Such an η will be a nonzero element of $K_0(ER)$.

Let S be the infinite cyclic group generated by t, written additively, and make S into a ring by requiring all products be zero. Thus $S = \mathbf{Z}[t]t/(t^2, t^3)$.

We construct the s.e.s. in *Ring* of canonical maps

$$(t^2, t^3)\mathbf{Z}[t] \to t\mathbf{Z}[t] \to S,$$

and let $R = (t^2, t^3)\mathbf{Z}[t]$. Thus R^+ is a cusp at the origin defined over $\mathbf{Z}$. Also let $F = t\mathbf{Z}[t]$. We have a commutative diagram with rows s.e.s.'s-

$$\begin{array}{ccccc} R & \longrightarrow & F & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ R[x] & \longrightarrow & F[x] & \longrightarrow & S[x] \end{array}$$

where vertical arrows are inclusions. By results of [**2**],

$$K_i(F) = 0 = K_i(F[x])$$

for $i=0$, 1. ($F^+ = \mathbf{Z}[t]$, $F[x] = E(\mathbf{Z}[x])$, and $\mathbf{Z}[x]$ is regular.) Thus, the exact sequence of [3, 5.8], yields a commutative diagram

$$\begin{array}{ccc} K_1(S) & \xrightarrow{\cong} & K_0(R) \\ \downarrow & & \downarrow \\ K_1(S[x]) & \xrightarrow{\cong} & K_0(R[x]) \end{array}$$

where horizontal arrows are isomorphisms. Thus, to construct an $\eta \in K_0(R[x])$ having the required properties, it is equivalent to construct $\xi \in K_1(S[x])$ such that $K_1(\partial_x^0)\xi = 0$, $K_1(\partial_x^1)\xi \neq 0$; $\partial_x^i : S[x] \to S$ is evaluation at i, $i=0$, 1.

Let ξ be the class of the unit $1+tx$ of $S[x]^+$ in $K_1(S[x])$. Using the determinant it is very easy now to verify that ξ has the required properties.

We have observed already that η is a nonzero element of $K_0(ER)$ such that $K_0(\partial_x^1)\eta \neq 0$. Consider now the commutative diagram of §2:

$$\begin{array}{ccc} \mathrm{GL}(ER[x, x^{-1}]) & \xrightarrow{\phi_{ER}} & K_0(ER) \\ \downarrow \mathrm{GL}(\partial_x^1) & & \downarrow K_0(\partial_x^1) \\ \mathrm{GL}(R[x, x^{-1}]) & \xrightarrow{\phi_R} & K_0(R) \end{array}$$

Since ϕ_{ER} is surjective, it follows that the composition $\phi_R \circ \mathrm{GL}(\partial_x^1) \neq 0$. But we saw that

$$\kappa_1^{\mathrm{GL}}(R[x, x^{-1}]) = \mathrm{GL}(R[x, x^{-1}])/\mathrm{Im}\ \mathrm{GL}(\partial_x^1).$$

It follows that ϕ_R does not factor through κ_1^{GL} for $R = (t^2, t^3)\mathbf{Z}[t]$.

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