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ON H-SPACES mod $\mathfrak{C}_p$

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ABSTRACT. We show that if a space X is an H-space mod $\mathfrak{C}_p$, then X is "dominated" by the reduced product space X_∞ $(=\Omega\Sigma X)$ of X and also by the component of the identity map in the space of all maps from X to X. We then deduce algebraic facts about the suspension homomorphism and the homomorphisms induced by the evaluation map. Making use of a new and pretty result of S. Weingram we give a short proof of the result of W. Browder that the mod p Hurewicz homomorphism is zero in even dimensions for an H-space mod $\mathfrak{C}_p$.

All spaces are assumed to be based and all maps and homotopies are to preserve base points, except in function spaces. Let $\mathfrak{C}_p$ (resp. $\mathfrak{T}_p$) denote the Serre class of finite (resp. torsion) abelian groups having no element with order a positive power of the prime p. A map $f: X \to Y$ is a $\mathfrak{C}_p$-equivalence if it induces an isomorphism of homology groups with coefficients Z_p. A space X is an H-space mod $\mathfrak{C}_p$ [6] if there is a map $X \times X \to X$ of type (ϕ, ϕ), where ϕ is a $\mathfrak{C}_p$-equivalence and is an h-space mod $\mathfrak{C}_p$ [1] if there is a map $X \times X \to X$ of type (ψ, ϕ), where ψ and ϕ are both $\mathfrak{C}_p$-equivalences. Since the Whitehead product $[2\iota_n, 2\iota_n]$ is zero, where ι_n generates $\pi_n(S^n)$ and n is odd, it follows that an odd dimensional sphere is an H-space mod $\mathfrak{C}_p$ for any odd prime p. Let X_∞ $(=\Omega\Sigma X)$ denote the reduced product space of X [5] and $L(X, Y; k)$ denote the path component of $k: X \to Y$ in the space of all (nonbased) maps from X to Y with k as base point; the compact-open topology is used throughout for function spaces.

THEOREM 1. *Let X be a* 1*-connected finite* CW*-complex and an H-space* mod $\mathfrak{C}_p$. *Then there is a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & L(X, X; 1) \\ {\scriptstyle\iota}\downarrow & \searrow^{\psi} & \downarrow{\scriptstyle\omega} \\ X_\infty & \longrightarrow & X \end{array}$$

where ι is the inclusion map, ω is the evaluation map, $\omega(f) = f(*)$, and

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ψ is a $\mathfrak{C}_p$-equivalence. It follows that

(i) *the kernel of the suspension homomorphism is, in each dimension, an element of $\mathfrak{C}_p$,*

(ii) *the evaluation map induces a $\mathfrak{C}_p$-epimorphism in homotopy (in the notation of* [2], *X is a G-space* mod $\mathfrak{C}_p$) *and homology in each dimension, and*

(iii) (*Browder* [**1**, *Theorem* 6.9])

$$0 = h \otimes 1 : \pi_{2n}(X) \otimes Z_p \to H_{2n}(X) \otimes Z_p$$

for all n, where h is the Hurewicz homomorphism, and $\pi_2(X) \in \mathfrak{C}_p$.

THEOREM 2. *Let X be a* 1*-connected finite* CW*-complex and an h-space* mod $\mathfrak{C}_p$. *Then* (ii) *and* (iii) (*above*) *hold.*

PROOF OF THEOREM 1. Let $\mu: X \times X \to X$ be a map of type (ϕ, ϕ), where ϕ is a $\mathfrak{C}_p$-equivalence and consider the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\nu} & \tilde{L}(X, X; 1) & \xrightarrow{\rho} & L(X, X; 1) & & \\ \gamma\downarrow & & \phi''\downarrow & & \phi'\downarrow & \searrow^{\omega} & \\ X & \xrightarrow{\mu''} & \tilde{L}(X, X; \phi) & \xrightarrow{\rho'} & L(X, X; \phi) & \xrightarrow{\omega} & X \end{array}$$

where ρ and ρ' are projections from universal covering spaces, ϕ' is defined by $\phi'(f) = f\phi$, ϕ'' is the lifting of ϕ' and μ'' is the lifting of the adjoint μ' of μ. By Lemma 1 (below) ϕ' induces a $\mathfrak{C}_p$-isomorphism in homotopy in all dimensions $\geqq 2$; thus ϕ'' induces a $\mathfrak{C}_p$-isomorphism in homotopy in all dimensions. Now X is p-universal [**6**, Theorem 4.2] (it is here that we need X to be an H-space mod $\mathfrak{C}_p$ as opposed to an h-space mod $\mathfrak{C}_p$) and so [**6**, Theorem 2.1] there are maps γ and ν making the above diagram (homotopy) commutative, where γ is a $\mathfrak{C}_p$-equivalence. Since ϕ is a based map, $\omega\phi' = \omega$ and $\psi \equiv \omega\rho\nu = \omega\rho'\mu''\gamma = \omega\mu'\gamma = \phi\gamma$ is a $\mathfrak{C}_p$-equivalence. We now obtain the desired diagram by applying Lemma 2 (below) to the map $\rho\nu$.

To prove (i) it suffices to show that ι induces a $\mathfrak{C}_p$-monomorphism in homotopy in each dimension since the suspension homomorphism is the following composite

$$\pi_n(X) \xrightarrow{\iota_*} \pi_n(X_\infty) = \pi_n(\Omega\Sigma X) = \pi_{n+1}(\Sigma X).$$

Since $\psi_* : H_n(X) \to H_n(X)$ is a $\mathfrak{C}_p$-isomorphism for each n and $\pi_n(X)$ is finitely generated for each n, it follows that ψ induces a $\mathfrak{C}_p$-isomorphism in homotopy in each dimension [**9**, Theorem 22, p. 512]; hence ι induces a $\mathfrak{C}_p$-monomorphism in homotopy in each dimension. The proof of (ii) is similar.

Assertion (iii) is a special case of Theorem 3 (below) since $L(X, X; 1)$ is an H-space (under composition) and has the homotopy type of a connected CW-complex [7].

Proof of Theorem 2. Let $\mu: X\times X\to X$ be a map of type (ψ, ϕ), where ψ and ϕ are both $\mathfrak{C}_p$-equivalences. Although we have been unable to obtain the diagram in Theorem 1, we still have the diagram in the proof of Theorem 1 except for the maps γ and ν and here $\omega\rho'\mu''=\psi$. The remainder of the proof is essentially the same as in Theorem 1.

Remark. We do not know if a G-space mod $\mathfrak{C}_p$ is an h-space mod $\mathfrak{C}_p$ (compare [2]). If the 1-connectedness condition is dropped then the answer is "no" [8].

Theorem 3. *Let Y be a connected* CW-*complex and an H-space, X be a finite* CW-*complex and $f: Y\to X$ be a map such that $f_*:\pi_{2n}(Y)\to\pi_{2n}(X)$ is a $\mathfrak{C}_p$-epimorphism. Then $h(\pi_{2n}(X))\in\mathfrak{C}_p$, where h is the Hurewicz homomorphism.*

This result (see [11, Theorem 2.1]) is an easy consequence of the following

Theorem [11, Theorem 1.7]. *Let $g:(S^{2n})_\infty\to K(G, 2n)$ be a nontrivial map, where $K(G, 2n)$ is an Eilenberg-Mac Lane space and G is a finitely generated abelian group. Then g is incompressible (i.e., does not factor through a finite* CW-*complex).*

Proof of Theorem 3. Assume that the assertion is false. Then there exists $\alpha\in\pi_{2n}(X)$ such that p divides the order of $h(\alpha)$. Consider the exact sequence

$$\pi_{2n}(Y)\xrightarrow{f_*}\pi_{2n}(X)\xrightarrow{q}\operatorname{coker}(f_*).$$

Choose $\beta\in\pi_{2n}(Y)$ so that $f_*(\beta)=r\alpha$, where r is the order of $q(\alpha)$. Since coker $(f_*)\in\mathfrak{C}_p$, r is prime to p and so $h(r\alpha)\neq 0$; thus there is a map $g:X\to K(G, 2n)$ such that $g(r\alpha)$ is nontrivial. Then the composite

$$(S^{2n})_\infty\xrightarrow{\beta_\infty}Y_\infty\xrightarrow{\rho}Y\xrightarrow{f}X\xrightarrow{g}K(G, 2n),$$

where β_∞ is the map induced by β and ρ is a retraction map ([5] and [4, p. 208]), is a nontrivial map and this contradicts Weingram's theorem which was stated above.

Lemma 1. *Let X be a 1-connected finite* CW-*complex and let $\phi: X\to X$ be a $\mathfrak{C}_p$-equivalence. Then*

$$\phi': L(X, X; 1) \to L(X, X; \phi), \qquad \phi'(f) = f\phi,$$

induces a $\mathfrak{C}_p$-isomorphism in homotopy in dimensions $\geqq 2$.

Proof. Let Y be the mapping cylinder of ϕ, $\iota: X \to Y$ and $i: X \to Y$ be the inclusion maps of X onto the domain and range of the mapping cylinder, respectively, and let $r: Y \to X$ be the retraction map onto the range of Y. Consider the diagram

$$\begin{array}{ccc} L(Y, Y; 1) & \xrightarrow{\alpha} & L(X, X; 1) \\ \iota' \downarrow & & \phi' \downarrow \\ L(X, Y; \iota) & \xrightarrow{\beta} & L(X, X; \phi) \end{array}$$

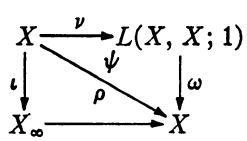
where $\iota'(f) = f\iota$, $\alpha(f) = rfi$ and $\beta(g) = rg$. It is easy to see that the diagram is homotopy commutative and that α and β are homotopy equivalences. Therefore, it suffices to consider the map ι' which is a fibration (since ι is a cofibration) with fiber F, $F = \{f: Y \to Y: f \simeq 1$ and $f | X = \iota\}$. Since the homotopy groups of each of the above function spaces are finitely generated abelian in dimensions $\geqq 2$ [**10**, Theorem 4], the result will follow if we show that $\pi_i(F) \in \mathfrak{T}_p$ for all $i \geqq 1$. Note that F is an H-space so that $\pi_1(F)$ is abelian.

Let $a: S^i \to F$ be a representative of $[a] \in \pi_i(F)$ and let $a': S^i \times Y \to Y$ be the adjoint of a. We extend a' to a map $a'': B^{i+1} \times X \cup S^i \times Y \to Y$, where $a'' | B^{i+1} \times X$ is projection onto X followed by ι. The obstructions to extending a'' to $B^{i+1} \times Y$ lie in

$$\begin{aligned} &H^{r+1}(B^{i+1} \times Y, B^{i+1} \times X \cup S^i \times Y; \pi_r(Y)) \\ &\qquad = H^{i+1}(B^{i+1}, S^i; Z) \otimes H^{r-i}(Y, X; \pi_r(Y)), \end{aligned}$$

which is an element of $\mathfrak{C}_p$ since ϕ and hence ι is a $\mathfrak{C}_p$-equivalence. By a straightforward argument one can show that there is a map $c: (B^{i+1}, S^i) \to (B^{i+1}, S^i)$ of degree s, with s prime to p, such that $a''(c \times 1)$ extends to $B^{i+1} \times Y$. The adjoint of this extension is a null homotopy of a representative of $s[a]$ and the result follows.

Lemma 2. *Let X be a locally compact space and $\nu: X \to L(X, X; 1)$ be a map such that $\omega\nu = \psi$. Then there is a commutative diagram*



Proof. Since ν is a based map and $L(X, X; 1)$ is an H-space it follows that ν extends to a map $\nu': X_\infty \to L(X, X; 1)$. Let $\rho = \omega\nu'$.

REMARK. Note that the conditions on the map ν in the above lemma are precisely that its adjoint be a map $X \times X \to X$ of type $(\psi, 1)$. What we have done above can be rephrased as follows: an H-space mod $\mathfrak{C}_p$ is an h-space mod $\mathfrak{C}_p$ having a "multiplication" with a one-sided unit. We have previously shown that an h-space mod $\mathfrak{F}$, where $\mathfrak{F}$ denotes the Serre class of finite abelian groups, has a multiplication with a one-sided unit [2]. Thus an h-space mod $\mathfrak{F}$ is an H-space mod $\mathfrak{F}$ and, moreover, the diagram in Theorem 1 can be obtained, where ψ is now an $\mathfrak{F}$-equivalence; that is, ψ induces an isomorphism of homology groups with rational coefficients. We do not know if an h-space mod $\mathfrak{C}_p$ is an H-space mod $\mathfrak{C}_p$.

REMARK. Let α be in the kernel of the suspension homomorphism (e.g. a HOWP in the sense of Porter) $\pi_m(S^n) \to \pi_{m+1}(S^{n+1})$, n odd. Since S^n is an H-space mod $\mathfrak{C}_p$ for each odd prime p, it follows from Theorem 1(i) that α has order 2^r for some r. In fact, $4\alpha = 0$: Since the Whitehead product $[2\iota_n, \iota_n]$ is zero, it follows from Lemma 2 that there is a map $\rho: (S^n)_\infty \to S^n$ such that $\rho\iota$ ($\iota: S^n \to (S^n)_\infty$ is the inclusion map) is a map of degree 2. Thus $0 = \rho_* \iota_*(\alpha) = 2\alpha + [\iota_n, \iota_n] \circ H_0(\alpha)$ (see [3] for the last equality); in particular $4\alpha = 0$.

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