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THE EQUIVALENCE OF THE LEAST UPPER BOUND PROPERTY AND THE HAHN-BANACH EXTENSION PROPERTY IN ORDERED LINEAR SPACES[1]

TING-ON TO

ABSTRACT. Let V be a partially ordered (real) linear space with the positive wedge C. It is known that V has the least upper bound property if and only if V has the Hahn-Banach extension property and C is lineally closed. In recent papers, W. E. Bonnice and R. J. Silverman proved that the Hahn-Banach extension and the least upper bound properties are equivalent. We found that their proof is valid only for a restricted class of partially ordered linear spaces. In the present paper, we supply a proof for the general case. We prove that if V contains a partially ordered linear subspace W of dimension $\geqq 2$, whose induced wedge $K = W \cap C$ satisfies $K \cup (-K) = W$ and $K \cap (-K) = \{\text{zero vector}\}$, then V fails to have the Hahn-Banach extension property. From this the desired result follows.

1. **Introduction.** In [1] W. E. Bonnice and R. J. Silverman proved a theorem which states that in a (partially) ordered (real) linear space the least upper bound property and the Hahn-Banach extension property are equivalent. We indicated in [3, p. 165] that their proof is only valid for a restricted class of ordered linear spaces. The purpose of this paper is to supply a proof for the general case.

2. **Definitions and preliminary lemmas.** In this paper, we consider linear spaces over the field R of real numbers. A nonempty subset C of a linear space V is said to be a wedge if $u, v \in C$ and $t \in R$, $t \geqq 0$, imply that $u+v$ and tu are in C. If V is ordered by a vector ordering $\geqq$, then the set $C = \{v : v \geqq \theta$, the zero element of $V\}$ is a wedge and is called the positive wedge of V determined by $\geqq$. Conversely, a wedge C in V determines a vector ordering $\geqq$ by taking $a \geqq b$ if and only if $a - b \in C$. Therefore a wedge C uniquely determines and is determined by a vector ordering $\geqq$.

A wedge C is said to be sharp if $u \in C$ and $-u \in C$ imply that

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$u=\theta$. It follows that the vector ordering $\geqq$ determined by the wedge C is antisymmetric (i.e., $a\geqq b\geqq a$ implies $a=b$) if and only if C is sharp.

Following [1], [2] we use the notation $(V;\ C)$ to denote a (partially) ordered linear space (OLS) V with the positive wedge C. An OLS $(V;\ C)$ is said to have the least upper bound property (LUBP) if every set of elements of V with an upper bound has a least upper bound (not necessarily unique). An OLS $(V;\ C)$ is said to have the Hahn-Banach extension property (HBEP) if given (1) a linear space X, (2) a linear subspace Y of X, (3) a function $p: X\rightarrow V$ which is sublinear (i.e., subadditive and positively homogeneous), and (4) a linear function $f\colon Y\rightarrow V$ such that $p(y)-f(y)\in C$ for all $y\in Y$, then there is a linear extension $F: X\rightarrow V$ of f such that $p(x)-F(x)\in C$ for all $x\in X$.

A wedge C in a linear space V is said to be lineally closed if the intersection of C with every line in V is a closed set in the natural topology of the line. A wedge C in a linear space V is said to be a semispace-wedge if $C\cup(-C)=V$ and $C\cap(-C)=\{\theta\}$. An OLS $(V;\ C)$ is said to be a lexicographically ordered linear space (LOLS) if the linear space V is of dimension $\geqq 2$ and if the positive wedge C is a semispace-wedge. It is clear that any ordered linear subspace of dimension $\geqq 2$ of a LOLS is still a LOLS.

Some preliminary lemmas concerning the above concepts are stated as follows:

LEMMA A [4, p. 9]. *Let $(V;\ C)$ be an OLS, X a linear space, Y a linear subspace of X such that the codimension of Y in X is* 1, *p a sublinear function from X into V, and let f be a linear function from Y into V such that $p(y)-f(y)\in C$ for all $y\in Y$. Then f has a linear extension F from X into V such that $p(x)-F(x)\in C$ for all $x\in X$ if and only if for each $x_0\in X\sim Y$ there exists $v_0\in V$ such that $-p(-y-x_0)-f(y)\leqq v_0 \leqq p(y'+x_0)-f(y')$ for every $y,\ y'\in Y$.*

LEMMA B ([4, p. 105], [5]). *An OLS $(V;\ C)$ has the LUBP if and only if $(V;\ C)$ has the HBEP and C is lineally closed.*

Therefore by Lemma B, the equivalence of the LUBP and the HBEP in an OLS $(V;\ C)$ will be established if we can show that the HBEP implies that the wedge C is lineally closed (we will do this in §3, Lemma 4). To this end, the following lemma will be used.

LEMMA C [1, pp. 844–845]. *If an OLS $(V;\ C)$ has the HBEP and if the positive wedge C is not lineally closed, then there exists a 2-dimensional lexicographically ordered linear subspace of $(V;\ C)$.*

3. **Main theorem.** We begin with the following lemmas.

LEMMA 1. *Let* $(V; C)$ *be an OLS. If there is a family* $\{(V_\xi; C_\xi)\}_{\xi\in E}$ *of ordered linear subspaces of* $(V; C)$, *where* E *is a nonempty set of indices ordered by a linear order* $\prec$, *satisfying the following conditions*:

(i) *the induced wedges* $C_\xi = V_\xi \cap C$, $\xi \in E$, *are semispace-wedges*;

(ii) *for every* $\xi, \eta \in E$, $C_\xi < (C_\eta \sim \{\theta\})^2$ *if and only if* $\xi \prec \eta$;

(iii) $V = \sum_{\xi\in E} V_\xi$, *the direct sum of* V_ξ, $\xi \in E$;

then C *is a semispace-wedge in* V.

PROOF. It is clear that for each nonzero element $v \in V$, v has a unique representation $v = \sum_{i=1}^{k} \lambda_{\xi_i} v_{\xi_i}$ for some $\lambda_{\xi_i} = 1$ or -1, $v_{\xi_i} \in C_{\xi_i} \sim \{\theta\}$, $\xi_i \in E$, $i = 1, 2, \cdots, k$, with $\xi_1 \prec \xi_2 \prec \cdots \prec \xi_k$.

Since $(C_{\xi_k} \sim \{\theta\}) > C_{\xi_i}$, $i = 1, 2, \cdots, k-1$,

$$v = v_{\xi_k} - \sum_{i=1}^{k-1} (-\lambda_{\xi_i}) v_{\xi_i} > \theta, \qquad \text{if } \lambda_{\xi_k} = 1;$$

and

$$v = \sum_{i=1}^{k-1} \lambda_{\xi_i} v_{\xi_i} - v_{\xi_k} < \theta, \qquad \text{if } \lambda_{\xi_k} = -1.$$

This shows that $V = C \cup (-C)$ and $C \cap (-C) = \{\theta\}$, and hence C is a semispace-wedge in V.

The following two corollaries follow immediately from the lemma.

COROLLARY 1.1. *Let* $(V; C)$ *be an OLS such that* C *is sharp. If there is a Hamel basis* $B = \{b_\xi\}_{\xi\in E}$ *of* V, *where* E *is a nonempty set of indices ordered by a linear order* $\succ$, *such that* $B > \theta$ *and such that* $\alpha b_\eta > \beta b_\xi$ *for all real numbers* $\alpha > 0$, $\beta > 0$, *and for every* $\eta, \xi \in E$ *with* $\eta \succ \xi$, *then* C *is a semispace-wedge in* V.

COROLLARY 1.2. *Let* $(V; C)$ *be an n-dimensional OLS such that* C *is sharp. If there is a basis* $\{b_i\}_{i=1}^n$ *of* V *such that* $\alpha_1 b_1 > \alpha_2 b_2 > \cdots > \alpha_n b_n > \theta$ *for all* $\alpha_i > 0$, $i = 1, 2, \cdots, n$, *then* C *is a semispace-wedge. Moreover,* C *can be expressed as the set*

$$\left\{ \sum_{i=1}^{n} \lambda_i b_i : \lambda_i \in R,\ i = 1, 2, \cdots, n, \textit{ and the first } \lambda_i \textit{ not to vanish is positive} \right\}.$$

The above corollary and its converse are well-known results [6], [7].

[2] Throughout $A \leqq B$ signifies that $a \leqq b$ for all $a \in A$ and $b \in B$; and $c \leqq A$ signifies that $c \leqq a$ for all $a \in A$.

LEMMA 2. *If $(V; C)$ is a LOLS, $\{v_i\}_{i=1}^n$ is a basis of an n-dimensional $(n \geqq 1)$ linear subspace V_n of V such that $\theta < \alpha_1 v_1 < \alpha_2 v_2 < \cdots < \alpha_n v_n$ for all $\alpha_i > 0$, $i = 1, 2, \cdots, n$, and V_{n+m} is an $(n+m)$-dimensional $(m \geqq 1)$ linear subspace of V containing V_n, then there exists a basis $\{v_1', v_2', \cdots, v_{n+m}'\}$ of V_{n+m} containing $\{v_i\}_{i=1}^n$ such that $\theta < \alpha_1 v_1' < \alpha_2 v_2' < \cdots < \alpha_{n+m} v_{n+m}'$ for all $\alpha_i > 0$, $i = 1, 2, \cdots, n+m$.*

The proof of Lemma 2, which is easily established by induction, is omitted.

Let $(V; C)$ be a LOLS. The wedge C is said to be a type (I) semispace-wedge if there exist $u_1, u \in C \sim \{\theta\}$ such that $u_1 < \alpha u$ for every $\alpha > 0$, and satisfying the following condition (I):

(I) There is no $v \in V$ such that $\alpha_1 u_1 < v < \alpha u$ for every $\alpha > 0$, $\alpha_1 > 0$.

C is said to be a type (II) semispace-wedge if it is not of type (I).

It is worth remarking that for the finite dimensional case, the positive wedge of any LOLS is a type (I) semispace-wedge but that for the infinite dimensional case, there exist many LOLS's each having its positive wedge which is a type (II) semispace-wedge.

The following lemma is the main result of this paper:

LEMMA 3. *Let $(V; C)$ be an OLS. If $(V; C)$ contains a lexicographically ordered linear subspace $(W; K)$, where $K = W \cap C$, then $(V; C)$ fails to have the HBEP.*

PROOF. We first assume that the positive wedge C is sharp.

A Zorn's Lemma argument guarantees the existence of a maximal lexicographically ordered linear subspace $(W^*; K^*)$ of $(V; C)$ containing $(W; K)$ where $K^* = W^* \cap C$. We shall show that if $(V; C)$ has the HBEP, then $(W^*; K^*)$ fails to be maximal. This contradiction will establish the lemma in case C is sharp. To this end, we assume that $(V; C)$ has the HBEP and consider the following two cases:

Case 1. K^* is a type (I) semispace-wedge.

Let $u_1, u \in K^* \sim \{\theta\}$, such that $u_1 < \alpha u$ for every $\alpha > 0$, and satisfying the condition (I), i.e., there is no $w \in W^*$ such that $\alpha_1 u_1 < w < \alpha u$ for every $\alpha_1 > 0$, $\alpha > 0$.

Let

$$X = \{(t_1, t_2) : t_i \in R, i = 1, 2\} = R_2, \qquad Y = \{(0, t_2) : t_2 \in R\}.$$

Define $P: X \to V$ by

$$p((t_1, t_2)) = -(t_1^+ t_2^+)^{1/2} u_1 + (|t_2| + t_1^+) u, \qquad \text{if } t_1 = t_2;$$

$$p((t_1, t_2)) = -(t_1^+ t_2^+)^{1/2} u_1 + \{|t_2| + t_1^+ - [(t_2^+ t_1^-)/(t_2 - t_1)]\} u,$$

$$\text{for all } t_1 \neq t_2,$$

where $t^+ = \sup\{t, 0\}$ and $t^- = \sup\{-t, 0\}$.

Define $f: Y \to V$ by $f((0, t_2)) = t_2 u$ for every $t_2 \in R$.

Then f is a linear function and $p(y) - f(y) \in K^*$ for every $y \in Y$ Moreover, referring to [2, p. 221, Case (2v) and p. 217, Example 2], p is a sublinear function from X into W^*. Note that

$$\begin{aligned} -p((-1, -t_2)) - f((0, t_2)) &= -2t_2 u, && \text{if } t_2 \geqq 0; \\ &= (-t_2/(1 - t_2))u, && \text{if } t_2 < 0; \end{aligned}$$

and

$$\begin{aligned} p((1, t_2')) - f((0, t_2')) &= -(t_2')^{1/2} u_1 + u, && \text{if } t_2' \geqq 0; \\ &= (-2t_2' + 1)u, && \text{if } t_2' < 0. \end{aligned}$$

Let

$$S = \{w \in W^* : w = (-t_2/(1 - t_2))u,\ t_2 \leqq 0\},$$
$$T = \{w \in W^* : w = -(t_2')^{1/2} u_1 + u,\ t_2' \geqq 0\}.$$

Then $S \leqq T$. Since $(V; C)$ has the HBEP, by Lemma A there exists $v_0 \in V$ such that $S \leqq v_0 \leqq T$. We claim that this element $v_0 \notin W^*$. Indeed, if there is $v_0 \in W^*$ such that $S \leqq v_0 \leqq T$, then $u - v_0 \in W^*$ and $(t_2')^{1/2} u_1 \leqq u - v_0 \leqq (1/(1 - t_2))u$ for all $t_2' > 0$ and $t_2 < 0$, a contradiction.

Let $v_0' = u - v_0$, then $\alpha_1 u_1 < v_0' < \alpha u$ for every $\alpha_1 > 0$, $\alpha > 0$, and $v_0' \notin W^*$. Let V' be the subspace of V spanned by W^* and v_0'.

We shall prove that the ordered linear subspace $(V'; C')$, where $C' = V' \cap C$, is a LOLS. Let $W_2 = \text{lin}\{u_1, u\}$,[3] $V_3' = \text{lin}\{u_1, v_0', u\}$. Then, by Corollary 1.2, $(W_2; K^* \cap W_2)$ and $(V_3'; C \cap V_3')$ are LOLS's.

Consider each nonzero element $w^* + \lambda v_0' \in V'$, where $w^* \in W^*$, $\lambda \in R$. If $w^* \in W_2$, then $w^* + \lambda v_0' \in V_3'$ and hence $w^* + \lambda v_0'$ belongs to one and only one of the wedges $C' \cap V_3'$ and $-C' \cap V_3'$. Thus, $w^* + \lambda v_0'$ belongs to one and only one of the wedges C' and $-C'$. If $w^* \notin W_2$, let $W_3 = \text{lin}\{u_1, u, w^*\}$ and let $V_4' = \text{lin}(W_3 \cup \{v_0'\})$. Since $(W_3; W_3 \cap K^*)$ is an ordered linear subspace of a LOLS $(W^*; K^*)$, by Lemma 2, there is $w_0^* \in W_3$ such that $\{u_1, u, w_0^*\}$ forms a basis for W_3 and is such that either $\theta < \alpha w_0^* < u_1 < \beta u$, or $\alpha u_1 < u < \beta w_0^*$, or $\alpha u_1 < w_0^* < \beta u$, for every $\alpha > 0$, $\beta > 0$. By our hypothesis, the last case is excluded. In the first case, we have $\theta < \alpha w_0^* < u_1 < \beta v_0' < u$ for every $\alpha > 0$, $\beta > 0$; in the second case, we have $\alpha u_1 < v_0' < \beta u < w_0^*$ for every $\alpha > 0$, $\beta > 0$. Thus, by Corollary 1.2, in both cases the ordered linear subspace $(V_4'; V_4' \cap C')$ is a LOLS and hence $w^* + \lambda v_0'$ belongs to one and only one of the wedges $V_4' \cap C'$ and $-(V_4' \cap C')$. Thus $w^* + \lambda v_0'$ belongs

[3] Throughout lin A signifies the linear hull of the set A.

to one and only one of the wedges C' and $-C'$. This shows that $V' = C' \cup (-C')$ and $C' \cap (-C') = \{\theta\}$. Thus, $(V'; C')$ is a LOLS with a semispace-wedge C' and hence $(W^*; K^*)$ is not a maximal lexicographically ordered linear subspace in $(V; C)$.

Case 2. K^* is a type (II) semispace-wedge.

Let $u_1, u \in K^* \sim \{\theta\}$ be such that $u_1 < \alpha u$ for every $\alpha > 0$ and let $U = \{w \in W^* : \beta u_1 < w < \alpha u$, for every $\alpha > 0, \beta > 0\}$. Since K^* is of type (II), U is nonempty and has no maximal element relative to the order of $(W^*; K^*)$. Moreover, by the maximal principle U contains a maximal linearly independent subset B. Let Y be the linear space spanned by B and u_1, and let X be the linear space spanned by Y and u. Then $\{u_1\} \cup B$ and $\{u_1\} \cup B \cup \{u\}$ are Hamel bases of Y and X, respectively. Thus, for each $x \in X$, there exists a finite subset $\{u_{\xi_i}\}_{i=1}^{k}$ of B (where k is some integer and $u_{\xi_1} < u_{\xi_2} < \cdots < u_{\xi_k}$ relative to the ordering of $(V; C)$), so that x has the unique representation, $x = t_1 u_1 + \sum_{i=1}^{k} t_{\xi_i} u_{\xi_i} + tu$, where t_1, t are real numbers and t_{ξ_i}, $i = 1, 2, \cdots, k$, are nonzero real numbers if $x \notin \operatorname{lin}\{u_1, u\}$; they are all zero if $x \in \operatorname{lin}\{u_1, u\}$.

Let $f: Y \to V$ be a linear function defined by

$$f(y) = t_1 u_1 + \sum_{i=1}^{k} t_{\xi_i} u_{\xi_i}, \qquad \text{for all } y = t_1 u_1 + \sum_{i=1}^{k} t_{\xi_i} u_{\xi_i} \in Y,$$

and let $p: X \to V$ be a function defined by

$$p(x) = p_1(x) + p_2(x) + p_3(x) u, \text{ for every } x = t_1 u_1 + \sum_{i=1}^{k} t_{\xi_i} u_{\xi_i} + tu \in X,$$

where

$$p_1(x) = \overset{+}{t_1} u_1 + \sum_{i=1}^{k-1} \overset{+}{t}_{\xi_i} u_{\xi_i};$$

$$p_2(x) = (\overset{+}{t}_{\xi_k} - \overset{-}{t}\,\overset{+}{t}_{\xi_k} / (-t + t_{\xi_k})) u_{\xi_k} \qquad \text{if } t \neq t_{\xi_k};$$

$$p_2(x) = \overset{+}{t}_{\xi_k} u_{\xi_k} \qquad \text{if } t = t_{\xi_k};$$

$$p_3(x) = p_3'((t_1, t)) = \overset{+}{t_1} + \overset{+}{t} - \overset{-}{t_1}\overset{+}{t} / (t - t_1), \qquad \text{if } t \neq t_1,$$

and

$$p_3(x) = p_3'((t_1, t)) = \overset{+}{t_1} + \overset{+}{t} \qquad \text{if } t = t_1;$$

where $t^+ = \sup\{t, 0\}$ and $t^- = \sup\{-t, 0\}$.

It follows that $p(y)-f(y)\in C$ for all $y\in Y$. We claim that p is sublinear from X into V. It is clear that $p_1(x)$, $p_2(x)$ and $p_3(x)$ are positively homogeneous and that $p_1(x)$ is also subadditive. Since $u_{\xi_k}<\alpha u$ for every $\alpha>0$, $u_{\xi_k}\in B$, it remains to show that p_3 is subadditive and that $p_2(x)+p_2(x')\geqq p_2(x+x')$ whenever $p_3(x)+p_3(x')=p_3(x+x')$ for every $x, x'\in X$. It requires detailed computations to show analytically that p_3 is subadditive. However, the graph of p_3' makes this obvious. Hence the computations will be omitted.

Also, from the graph of p_3', it is clear that $p_3(x+x')=p_3(x)+p_3(x')$, where

$$x = t_1u_1 + \sum_{i=1}^{k} t_{\xi_i}u_{\xi_i} + tu, \qquad x' = t_1'u_1 + \sum_{i=1}^{k'} t_{\eta_i}'u_{\eta_i} + t'u,$$

if and only if

(a) $t_1=rt_1'$ and $t=rt'$, $r>0$, or
(b) $t_1\geqq 0, t\geqq 0, t_1'\geqq 0, t'\geqq 0$, or
(c) $t_1\geqq 0, t\leqq 0, t_1'\geqq 0, t'\leqq 0$, or
(d) $t_1\leqq 0, t\leqq 0, t_1'\leqq 0, t'\leqq 0$.

Simple direct computations show that in these cases $p_2(x+x')\leqq p_2(x)+p_2(x')$. Thus, p is sublinear from X into V. Furthermore,

$$-p(ru_{\xi_k}-u)-f(-ru_{\xi_k}) = (r/(1+r))u_{\xi_k}, \quad \text{for every } u_{\xi_k}\in B,\ r>0;$$

and

$$p(-su_1+u)-f(-su_1) = (1/(1+s))u+su_1, \qquad \text{for all } s\geqq 0.$$

Let $M=\{(r/(1+r))u_{\xi_k}:u_{\xi_k}\in B,\ r>0\}$ and let $N=\{(1/(1+s))u+su_1:s\geqq 0\}$. Then $M<N$. By Lemma A, since $(V;C)$ has the HBEP, there is $v_0\in V$ such that $M\leqq v_0\leqq N$.

We claim that this element $v_0\notin W^*$. For suppose on the contrary that $v_0\in W^*$, then from $(r/(1+r))u_{\xi_k}\leqq v_0\leqq(1/(1+s))u+su_1$, for every $r>0$, $s>0$ and $u_{\xi_k}\in B$, we have (i) $u, v_0\in K^*\sim\{\theta\}$ with $v_0<\alpha u$ for all $\alpha>0$, and (ii) for every $u'\in U$ there exists $\alpha'>0$ such that $\alpha'u'<v_0$. Clearly, (i) and (ii) are contradictory since K^* is a type (II) semi-space-wedge.

In order to show that $(W^*;K^*)$ is not maximal, let $V'=\text{lin}(W^*\cup\{v_0\})$. We claim that $(V';C')$, where $C'=V'\cap C$, is a LOLS. Let $W'=\text{lin}(B\cup\{u_1\})$, $W''=\text{lin}(B\cup\{u_1, u\})$, $V''=\text{lin}(W''\cup\{v_0\})$. Then $(W'; W'\cap K^*)$ and $(W''; W''\cap K^*)$ are lexicographically ordered linear subspaces of $(W^*;K^*)$. Furthermore, the ordered linear subspace $(V''; V''\cap C)$ of $(V;C)$ is a LOLS. For, if $u_\xi\in B$ we can choose $u'\in U$ such that $\lambda u_\xi<\alpha u'<\beta u$ for all $\lambda>0$, $\alpha>0$, $\beta>0$.

Since $u' \in U$ and $\frac{1}{2}u_\xi < v_0$ for all $u_\xi \in B$, there exists $\alpha' > 0$ such that $\alpha' u' < v_0$. Also, $M \leqq v_0 \leqq N$ implies that $v_0 < \beta u$ for all $\beta > 0$. Thus, $\lambda u_\xi < v_0 < \beta u$ for all $\lambda > 0$, $\beta > 0$. It follows that $W' \cap K^* < \alpha v_0 < \beta u$ for all $\alpha > 0$, $\beta > 0$. Therefore, by Lemma 1, $(V''; V'' \cap C)$ is a LOLS as was asserted.

To see that C' is a semispace-wedge in V', we consider each nonzero element $v = w^* + \lambda v_0 \in V'$, where $w^* \in W^*$, $\lambda \in R$. If $w^* \in W''$, then $v \in V''$ and hence v belongs to one and only one of the wedges $V'' \cap C'$ and $-(V'' \cap C')$. Thus, v belongs to one and only one of the wedges C' and $-C'$. If $w^* \notin W''$, let $W''' = \text{lin}(W'' \cup \{w^*\})$, $V''' = \text{lin}(W''' \cup \{v_0\})$ and let $W_3^* = \text{lin}\{u_1, u, w^*\}$.

Note that $(W_3^*; W_3^* \cap K^*)$ is a lexicographically ordered linear subspace of $(W^*; K^*)$; hence by Lemma 2 there is $w_0^* \in W_3^*$ such that $\{u_1, u, w_0^*\}$ forms a basis for W_3^* and is such that either $\theta < \alpha w_0^* < u_1 < \beta u$, or $\alpha u_1 < u < \beta w_0^*$, or $\alpha u_1 < w_0^* < \beta u$, for every $\alpha > 0$, $\beta > 0$. Since $w^* \notin W''$, $w_0^* \notin W''$ and hence $w_0^* \notin \text{lin } B$. Thus, the last case $\alpha u_1 < w_0^* < \beta u$, for every $\alpha > 0$, $\beta > 0$, is excluded. From the first case or the second case, we have $\theta < \alpha w_0^* < u_1 < C_B < \beta v_0 < u$ or $\alpha u_1 < C_B < \beta v_0 < u < \gamma w_0^*$, for every $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, respectively, where $C_B = \lambda B$, $\lambda > 0$. Therefore, by Lemma 1, in both cases the induced wedge $C''' = V''' \cap C'$ is a semispace-wedge in V''' and hence $v = w^* + \lambda v_0$ belongs to one and only one of the wedges C''' and $-C'''$. Thus, v belongs to one and only one of the wedges C' and $-C'$. This shows that $(V'; C')$ is a LOLS, and hence that $(W^*; K^*)$ is not maximal. This completes the proof of the lemma in case C is sharp.

In case C is not sharp, let $V_0 = \{v \in V : v \in C \cap -C\}$ and let $C_0 = V_0 \cap C$. Since any lexicographical order is antisymmetric, $W \cap V_0 = \{\theta\}$. Therefore, there exists a subspace V_1 of V containing W which is complementary to V_0 in V. Let $C_1 = V_1 \cap C$. Then C_1 is sharp. Thus, by the result that we have just proved, (V_1, C_1) fails to have the HBEP. Also, it is easy to verify that $(V; C)$ has the HBEP if and only if (V_1, C_1) has the HBEP. Therefore, $(V; C)$ fails to have the HBEP.

LEMMA 4. *If an OLS $(V; C)$ has the HBEP, then the positive wedge C is lineally closed.*

PROOF. Assume that $(V; C)$ has the HBEP and that the positive wedge C is not lineally closed. Then, by Lemma C, there exists a 2-dimensional lexicographically ordered linear subspace of $(V; C)$, and hence, by Lemma 3, $(V; C)$ fails to have the HBEP, a contradiction.

From Lemma B and Lemma 4, the following theorem is immediate:

THEOREM. *Let $(V; C)$ be an OLS. Then $(V; C)$ has the LUBP if and only if $(V; C)$ has the HBEP.*

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UNIVERSITY OF SASKATCHEWAN, SASKATOON, SASKATCHEWAN, CANADA

UNIVERSITY OF CANTERBURY, CHRISTCHURCH, NEW ZEALAND