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DECOMPOSITIONS OF FINITELY GENERATED MODULES

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ABSTRACT. A commutative ring with unit is called a d-ring if every finitely generated Loewy module is a direct sum of cyclic submodules. It is shown that every d-ring is a T-ring, i.e., Loewy modules over such rings satisfy a primary decomposition theorem. Some applications of this result are given.

1. **Introduction and notation.** All rings in this note are commutative with unit and modules are unital. The problem of characterizing those rings whose finitely generated torsion modules (in the sense that every element has nonzero order ideal) are direct sums of cyclic submodules appears to be difficult. Call such a ring a D-ring. E. Matlis has characterized the Noetherian domains which are D-rings as just the Dedekind domains [4]. Related results can be found in an article of R. B. Warfield, Jr. [5]. If one uses the term "torsion module" in the sense of S. E. Dickson [2], then torsion modules are just Loewy modules and the problem mentioned above becomes that of characterizing d-rings. In this note we shall obtain a characterization of such rings as a consequence of the fact that d-rings are, in the terminology of S. E. Dickson [2], T-rings. Another consequence of this fact is that extensive information about the structure of Loewy modules over d-rings can be obtained.

I would like to thank the referee for pointing out to me that a countability assumption originally imposed on the rings of this note could be eliminated.

The socle of an R-module M is the sum of all irreducible submodules of M and is denoted by $\mathrm{Soc}(M)$. If I is a maximal ideal of R, the *I-socle* of M, $\mathrm{Soc}(M, I)$, is the sum of all irreducible submodules of M which are annihilated by I. The ascending Loewy series of M, namely,

$$0 = L_0(M) \subseteq L_1(M) \subseteq \cdots \subseteq L_\lambda(M) = L_{\lambda+1}(M),$$

is defined inductively: If the submodules $L_\alpha(M)$ are defined for all $\alpha<\beta$ and β is a limit ordinal, then $L_\beta(M)$ is the union of all the pre-

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ceding terms. Otherwise $L_\beta(M)$ is the inverse image in M of $\mathrm{Soc}(M/L_{\beta-1}(M))$. The Loewy submodule of M, $L(M)$, is just $L_\lambda(M)$, where λ is such that $L_\lambda(M)=L_{\lambda+1}(M)$. If $L(M)=M$, then M is called a Loewy module. Analogously we define $L_\beta(M, I)$, $L(M, I)$ and I-Loewy modules. The submodule $L(M, I)$ is called the *I-Loewy submodule* of M. If every Loewy R-module M is the direct sum of its submodules $L(M, I)$ then, R is called a T-ring.

LEMMA 1. *Every Noetherian Loewy R-module is a direct sum of cyclic submodules if and only if R satisfies the condition*:

(′) *For every maximal ideal I of R, I/I^2 is trivial or irreducible.*

PROOF. If R satisfies (′) and M is a Noetherian R-module, let K be the annihilator of M in R. Then R/K is a Noetherian ring which satisfies (′). Hence M is a direct sum of cyclic R/K-submodules by Theorem 4 of [**5**, p. 170], so that M is a direct sum of R-submodules. On the other hand if there is a maximal ideal I of R such that I/I^2 is reducible, then we can construct (just as in the proof of Theorem 2 of [**5**, p. 168]) an indecomposable Noetherian R-module M which cannot be generated by fewer than two elements. Thus M is not a direct sum of cyclic submodules.

LEMMA 2. *If R satisfies (′) then every I-Loewy cyclic R-module is Noetherian.*

PROOF. First note that if n is a positive integer, M an I-Loewy R-module and $x\in M$, then $x\in L_n(M, I)$ if and only if $xI^n=0$. Also a simple induction on the I-Loewy series of M shows that if $x\in L(M, I)$ and $t\in I$, then there is an integer n such that $xt^n=0$.

Now suppose that R satisfies (′) and has a non-Noetherian cyclic I-Loewy module. Then there is an ideal K of R such that R/K is a non-Noetherian I-Loewy R-module. Replace R by R/K and we may assume that $K=0$, i.e., R is a non-Noetherian I-Loewy module. The ideal I is nil by the preceding paragraph. Furthermore it is easily seen that the only ideals of R between I and I^n (n any integer) are powers of I, since I/I^2 is irreducible. If there is an integer n such that $I^n=I^{n+1}$, then every element of $L_\omega(I, I)$ is annihilated by I^n. It follows that $L_n(I, I)=L_{n+1}(I, I)=I$. Therefore $I^{n+1}=0$ and R is Noetherian.

On the other hand if $I^{n+1}\subset I^n$ for all integers n, let $P=\bigcap_{n=1}^\infty I^n$. Then P is actually a prime ideal of R; for if $x, y\in R-P$, say $x\in I^r-I^{r+1}$ and $y\in I^s-I^{s+1}$, then $xR=I^r \bmod I^{r+s+1}$ and $yR=I^s \bmod I^{r+s+1}$. Thus $xyR=I^{r+s} \bmod I^{r+s+1}$ and $xy\notin I^{r+s+1}$. But I is a nil ideal of R,

so $I \subseteq P$ which is a contradiction. Consequently there is an integer n such that $I^n = I^{n+1}$ and the lemma follows.

Let us call the ring R a *special ring* if R is a non-Noetherian semiprime ring such that $\mathrm{Soc}(R)$ is a maximal ideal of R. The existence of special rings is the reason why a ring need not be a T-ring.

LEMMA 3. *The ring R is not a T-ring if and only if R has an ideal K such that R/K is a special ring.*

PROOF. Suppose that R is not a T-ring. Then there is a maximal ideal I_0 of R such that $\mathrm{Ext}_R(R/I_0, \oplus \sum_{I \neq I_0} (R/I)) \neq 0$, by Lemma 2.1 of [**2**, p. 350]. (Here the sum is a weak direct sum.) So let M be a nontrivial extension of $\oplus \sum_{I \neq I_0} (R/I) = N$ by R/I_0. Choose an element $m \in M - N$ and let K be the annihilator of m in R. Then R/K is R-isomorphic to mR. The socle of mR is not Noetherian; for in the contrary case mR is a direct sum of various I-Loewy submodules. This is obviously also the case for cyclic submodules of N. Since $M = N + mR$ is the sum of its finitely generated submodules, it would follow that M is a direct sum of its I-Loewy submodules. Therefore M would be a split extension of N by R/I_0, a contradiction. It follows that the socle of R/K is a non-Noetherian (R/K)-module. Furthermore $(R/K)/\mathrm{Soc}(R/K)$ is R-isomorphic to R/I_0 and hence irreducible, so $\mathrm{Soc}(R/K)$ is a maximal ideal of R/K. We may write $L = \mathrm{Soc}(R/K) = \oplus \sum_{i \in J} A_i$ where each A_i is a minimal ideal of R/K and J is an infinite index set. Note that if $i \in J$ and $A_i^2 = 0$, then $LA_i = 0$ and therefore $I_0 A_i = 0$. Since A_i is irreducible, A_i is R-isomorphic to R/I_0, contradicting the fact that A_i is R-isomorphic to a submodule of N. Hence $A_i^2 = A_i$ and there is an idempotent $e_i \in A_i$ such that $e_i(R/K) = A_i$. It follows that no nonzero element of $\mathrm{Soc}(R/K)$ is nilpotent and R/K is semiprime. Consequently R/K is a special ring.

Conversely suppose K is an ideal of R such that R/K is a special ring. It follows readily that $\mathrm{Soc}(R/K)$ is a direct sum of minimal ideals A_i, $i \in J$, where J is an infinite index set. Also there are idempotents e_i such that $A_i = e_i R$ for all $i \in J$. Let B_i be the anihilator of e_i in R/K. Then B_i is a maximal ideal of R/K containing $\sum_{j \neq i} A_j$. Hence if $i \neq j$, then $B_i \neq B_j$. Also $A_i \subseteq \mathrm{Soc}(R/K, B_i)$ for all $i \in J$. Clearly R/K is a Loewy R-module and if C_i is the inverse image in R of B_i under the natural homomorphism,then $C_i \neq C_j$ for $i \neq j$ and $\mathrm{Soc}(R/K, B_i) = \mathrm{Soc}(R/K, C_i)$. If R/K were a direct sum of its I-Loewy submodules, then the sum would involve a finite number of maximal ideals I of R, since R/K is a cyclic R-module. Thus $\mathrm{Soc}(R/K, I) \neq 0$ for only a finite number of maximal I, a contradiction. Hence R is not a T-ring.

LEMMA 4. *If R is a special ring whose socle is a direct sum of minimal ideals A_i, $i \in J$, and $r \in R - \mathrm{Soc}(R)$, then rR contains all but a finite number of the ideals A_i.*

PROOF. We have $\mathrm{Soc}(R) = I$ is a maximal ideal of R and (since R is semiprime) there are idempotents e_i such that $A_i = e_i R$. So there is an element $s \in R$ such that $rs = 1 \mod I$, i.e., for some $e_{i_1}, \cdots, e_{i_n}$, and elements $r_1, \cdots, r_n$ of R, $rs = 1 + \sum_j e_{i_j} r_j$. Hence for k distinct from the i_j's, $rse_k = e_k \in rR$.

PROPOSITION 1. *Every special ring is not a d-ring.*

PROOF. Let R be a special ring with socle $I_0 = \sum_{i \in J} A_i$, where each A_i is irreducible and $A_i^2 = A_i$. Write J as the disjoint union of three infinite sets J_k, $k = 1, 2, 3$. Let $S_k = \sum_{i \in J_k} A_i$, $i = 1, 2, 3$. Let F be a free R-module on the generators a and b and let K be the submodule $aS_1 + bS_2 + (a+b)S_3$. Define $M = F/K$; then M is not a direct sum of cyclic modules. To see this let $\alpha(x) = \{i \in J \mid xA_i = 0\}$ for each $x \in M$. Also call subsets of J almost equal if their symmetric difference is finite. Thus if x and y are in M and $x - y \in MI_0$, then $\alpha(x)$ and $\alpha(y)$ are almost equal. Also $\alpha(x)$ is almost equal to $\alpha(xr)$ for all $x \in M$ and $r \in R - I_0$ by Lemma 4. Suppose that M can be written as a direct sum of cyclic submodules C_i, $i = 1, \cdots, r$. Note that $M/MI_0 \cong F/FI_0$, so M/MI_0 is a two dimensional vector space over R/I_0. Hence there are exactly two C_i such that $C_i/C_iI_0 \cong R/I_0$. W.l.o.g. these are C_1 and C_2, so that $C_i \subseteq MI_0$ for $i > 2$. Let x_1 and x_2 be generators of C_1 and C_2. If r and s are in $R - I_0$, then $\alpha(x_1 r + x_2 s) = \alpha(x_1) \cap \alpha(x_2)$. If only one of r and s is in $R - I_0$, then $\alpha(x_1 r + x_2 s)$ is almost equal to $\alpha(x_1)$ or $\alpha(x_2)$. Thus if $x \in M - MI_0$, then $\alpha(x)$ is almost equal to $\alpha(x_1)$, $\alpha(x_2)$ or $\alpha(x_1) \cap \alpha(x_2)$. Let the images of a and b in M be denoted a^* and b^*; then M is generated by these elements, so neither α^*, b^* nor $a^* + b^*$ belongs to MI_0 (for M/MI_0 is two dimensional). But $\alpha(a^*) = J_1$, $\beta(b^*) = J_2$ and $\alpha(a^* + b^*) = J_3$; no two of these sets are almost equal and no one of them is almost equal to the intersection of the other two, contradicting the fact that each is almost equal to $\alpha(x_1)$, $\alpha(x_2)$ or $\alpha(x_1) \cap \alpha(x_2)$. Thus R is not a d-ring.

Proposition 1 and Lemma 3 yield:

THEOREM 1. *Every d-ring is a T-ring.*

3. **Applications.** The property of being a d-ring can be expressed in ideal theoretic terms:

THEOREM 2. *The following are equivalent conditions on the ring R:*
(1) *R is a d-ring.*

(2) *R is a T-ring satisfying condition* (′).

(3) *R satisfies condition* (′) *and has no ideals K such that R/K is a special ring.*

PROOF. The equivalence of (2) and (3) follows from Lemma 3. That (1) implies (2) follows from Theorem 1 and Lemma 1. Conversely if R satisfies (2) and M is a finitely generated Loewy R-module, then M is a direct sum of finitely generated I-Loewy submodules since R is a T-ring. Each of these submodules is Noetherian by Lemma 2. Hence M is a Noetherian R-module. Since R satisfies (′), M is a direct sum of cyclic submodules by Lemma 1. Therefore R is a d-ring.

COROLLARY. *Let R be a d-ring and M a Loewy R-module. Then $M=L_\omega(M)$ and if M is finitely generated, M is both Artinian and Noetherian.*

PROOF. By the proof of Theorem 2 any finitely generated Loewy R-module is Noetherian. Hence if $m\in M$, then $mR\subseteq L_n(M)$ for some integer n and $M=L_\omega(M)$. Also a Noetherian Loewy module M is Artinian since Noetherian socles are Artinian and $M=L_n(M)$ for some integer n.

REMARK. There does exist for each ordinal α a (commutative) ring whose Loewy modules can have Loewy length α. For details see L. Fuchs' article [**3**].

EXAMPLES. Each part of condition (3) of Theorem 2 is not alone sufficient to ensure that R is a d-ring. Let R be the ring of upper triangular 4×4 matrices over a field F with constant values along the diagonal, arbitrary values in the (1, 2)th and (3, 4)th entries and zeros elsewhere. Then R is a local ring with maximal ideal I such that $I^2=0$ and I/I^2 is reducible. Also R is Noetherian and a T-ring by Lemma 3. But by Lemma 1 there are Noetherian R-modules which are not direct sums of cyclic submodules. Hence R is not a d-ring.

On the other hand the property of being a d-ring is more restrictive than that of satisfying (′). To see this, note that any special ring R is (Von Neumann) regular. For if $a\in\mathrm{Soc}(R)$, it is clear that there exists an element $r\in R$ such that $a^2r=a$. If $a\in R-\mathrm{Soc}(R)$, pick an element $r\in R$ such that $1-ar=t\in\mathrm{Soc}(R)$. Then $a=a^2+at$. Since $at\in\mathrm{Soc}(R)$, we have $(at)R=(at)^2R$ and hence $at\in a^2R$, say $at=a^2u$. Then $a=a^2(r+u)$ and R is regular. Hence $I^2=I$ for every maximal ideal I of R, so that R satisfies (′). Yet R is not a d-ring by the equivalence of (1) and (3). Examples of special rings are easy to construct: Let F be any field, A an infinite weak direct sum of F-submodules $A_i\cong F$, $i\in J$. Then A is an algebra over F with coordinatewise multi-

plication. Now let R be the ring whose underlying set is the Cartesian product $A \times F$ with coordinatewise addition and multiplication given by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + a_1b_2 + a_2b_1, b_1b_2).$$

It is easy to see that R is a special ring whose socle is (the image of) A.

Let us call the ring R a *Loewy* ring in case R is a Loewy module over itself. Loewy D-rings can be completely classified:

THEOREM 3. *The Loewy ring R is a D-ring if and only if R is an Artinian PIR, and in this case every R-module is a direct sum of cyclic submodules.*

PROOF. Any module over an Artinian PIR is a direct sum of cyclic submodules by a result of I. Cohen and I. Kaplansky [**1**]. Conversely if R is a Loewy D-ring, then R is a cyclic Loewy module over itself and therefore Noetherian by the corollary to Theorem 2. Furthermore R satisfies (′) by Lemma 1. Hence R is a finite direct sum of special PIR's and Dedekind domains by Theorem 4 of [**5**, p. 170]. Since each of these Dedekind domains is a Loewy module over itself, R is a sum of only special PIR's and the result follows.

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