PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 3, November 1971

ON UNIVALENT FUNCTIONS WITH TWO PREASSIGNED VALUES

MAXWELL O. READE[1] AND ELIGIUSZ J. ZŁOTKIEWICZ[2]

ABSTRACT. Let $\mathfrak{M}_M$ denote the class of functions analytic and univalent in the unit disc Δ subject to the conditions

$$f(0) = 0, \qquad f(z_0) = z_0, \qquad |f(z)| < M,$$

where z_0, $z_0 \neq 0$, is a fixed point of Δ and $1 \leqq M \leqq \infty$.

In the present note, we determine by the method of circular symmetrization, the exact value of the "Koebe constant" for the class $\mathfrak{M}_M$. We also determine Koebe sets for the class $\mathfrak{M}_M^*$ consisting of the starlike functions, and for $\mathfrak{M}_M^\alpha$, consisting of all functions mapping Δ onto domains convex in the direction $e^{i\alpha}$.

By "Koebe set" we understand the set $\mathcal{K}(\mathfrak{M}_M)$, $\mathcal{K}(\mathfrak{M}_M) = \bigcap_{f \in \mathfrak{M}_M} f(\Delta)$.

1. **Introduction.** Let $\mathfrak{M}_M$ denote the class of functions analytic and univalent in the unit disc Δ subject to the conditions

$$f(0) = 0, \qquad f(z_0) = z_0, \qquad |f(z)| < M \tag{1}$$

where z_0, $z_0 \neq 0$, is a fixed point of Δ and $1 < M < \infty$.

So far as we know there are no results concerning class $\mathfrak{M}_M$, whereas many authors have considered certain extremal problems in the case $M = \infty$.

In the present note, we determine by the method of the circular symmetrization, the exact value of the Koebe constant for the class $\mathfrak{M}_M$. We also determine Koebe sets for the class $\mathfrak{M}_M^*$, consisting of the starlike functions, and for $\mathfrak{M}_\infty^\alpha$, consisting of all functions mapping Δ onto domains convex in the direction $e^{i\alpha}$.

By "Koebe set" [1] we understand the set $\mathcal{K}(\mathfrak{M}_M)$, $\mathcal{K}(\mathfrak{M}_M) = \bigcap_{f \in \mathfrak{M}_M} f(\Delta)$.

2. **Main results.** 2.1. We start with the determination of the Koebe constant for the class $\mathfrak{M}_M$.

Presented to the Society, November 28, 1970; received by the editors July 27, 1970.

AMS 1969 *subject classifications*. Primary 3042, 3052.

Key words and phrases. Univalent function, Koebe constant, Koebe domain, starlike mapping, convex in one direction, hyperbolic measure, Green's function, bounded functions.

[1] The first author acknowledges support received under National Science Foundation Grant GP-11158.

[2] The second author acknowledges support received from I.R.E.X.

THEOREM 1. *The image-domain of* Δ *under each function of the class* $\mathfrak{M}_M$ *contains the disc* $|w|<r(M)$ *where*

$$r(M) = 2\delta^2 - M - 2\delta(\delta^2 - M)^{1/2},$$
$$\delta = (M - |z_0|)(1 - |z_0|)^{-1}.$$

The number $r(M)$ cannot be replaced by any greater number without additional assumptions on the functions of the class $\mathfrak{M}_M$.

PROOF. Suppose that $\Omega = f(\Delta)$ and $\rho(a, b, \Omega)$ denotes the hyperbolic distance of the points a, b with respect to the domain Ω.

Since the hyperbolic distance is a conformal invariant, we have

$$\text{arc th } |z_0| = \rho(0, z_0, f(\Delta)), \qquad f \in \mathfrak{M}_M. \tag{2}$$

Let Ω^* be a domain obtained from Ω by using circular symmetrization with respect to the ray $[0, z_0, \infty)$. It is clear that the origin and z_0 are in Ω^*. Moreover, it is well known that circular symmetrization decreases the hyperbolic distance, so that

$$\rho(0, z_0, \Omega) \geqq \rho(0, z_0, \Omega^*) \tag{3}$$

holds. However, each domain Ω^* is contained in a domain D_h, $D_h = \Delta_M \setminus (-Me^{i\alpha}, -he^{i\alpha}]$ for some h, $h>0$, where Δ_M is the disc $|w| < M$. Hence

$$\rho(0, z_0, \Omega^*) \geqq \rho(0, z_0, D_h). \tag{4}$$

The conditions (2), (3) and (4) now yield our basic inequality

$$\text{arc th } |z_0| \geqq \rho(0, z_0, D_h). \tag{5}$$

For our purpose it is sufficient to consider $z_0 = r$. Now let $\varphi(z)$, $\varphi(0) = 0$ be the function that maps the domain D_h, which is now $\Delta_M \setminus (-M, -h]$ conformally onto the unit disc. Then we have

$$\rho(0, z_0, D_h) = \text{arc th } |\varphi(r)|,$$

where

$$\varphi(r) = \frac{2Mr}{d(r^2 + M^2) + 2Mr(1 - d) + ([d(M - r)^2 + 2Mr]^2 - 4M^2r^2)^{1/2}},$$
$$d = 4hM(M + h)^{-2}.$$

Thus (5) yields the inequality

$$d(M - r)^2 + 2Mr + ([d(M - r)^2 + 2Mr]^2 - 4M^2r^2)^{1/2} \geqq 2M,$$

which holds if

$$d \geqq M(1 - r)^2(M - r)^{-2}$$

or

$$h \geqq 2\delta^2 - M - 2\delta(\delta^2 - M)^{1/2}, \qquad \delta = (M - r)(1 - r)^{-1}.$$

The sign of equality in these last considerations occurs for the function $f(z)$ given by the equation

$$\frac{f(z)}{[M - f(z)]^2} = \left(\frac{1 - r}{M - r}\right)^2 \frac{z}{(1 - z)^2}.$$

This completes our proof.

COROLLARY 1. *For the cases $M = \infty$ and $z_0 = 0$ we have*

$$r(\infty) = \tfrac{1}{4}(1 - |z_0|)^2, \qquad r_0(M) = 2M^2 - M - 2M(M^2 - M)^{1/2},$$

respectively. Both constants are well known ([**3**], [**4**], *respectively*).

2.2. Now we shall be concerned with the determination of Koebe sets for starlike functions of the class $\mathfrak{M}_M$. Let Δ_M be the disc $\{w: |w| < M\}$ and let G be the family of all its subdomains that are (i) starshaped with respect to the origin and (ii) contain the fixed point z_0. For $D \in G$, let $g(w, z_0, D)$ be the Green's function with pole at z_0, and let $\mu(w_0)$ be defined as a solution of the following extremal problem:

$$\mu(w_0) = \operatorname*{l.u.b.}_{D \in G; w_0 \in \Delta_M \setminus D} g(0, z_0, D). \tag{6}$$

The following formula for $\mathcal{K}(\mathfrak{M}_M)$ was obtained by Krzyż and Złotkiewicz [**2**]:

$$\mathcal{K}(\mathfrak{M}_M) = \{w : \mu(w) < -\log |z_0|\}. \tag{7}$$

Hence, in order to determine $\mathcal{K}(\mathfrak{M}_M)$ it is sufficient to solve the extremal problem (6). We shall use (7) for the two classes $\mathfrak{M}_M^*$, $\mathfrak{M}_\infty^\alpha$ consisting of all starlike functions of the class $\mathfrak{M}_M$ and of those functions mapping Δ onto domains convex in the given direction $e^{i\alpha}$, respectively.

THEOREM 2. *The Koebe set of the class $\mathfrak{M}_M^*$ is determined by the condition*

$$|w - z_0| \frac{|M - w\bar{z}_0|}{(M + |w|)^2} + \left(\frac{Mw + z_0|w|}{M + |w_0|}\right)^2 \frac{1}{|w|} < \tfrac{1}{2}(1 + |z_0|^2). \tag{8}$$

The limit case, $M = \infty$, is

$$|w - z_0| + |w| < \tfrac{1}{2}(1 + |z_0|^2). \tag{9}$$

PROOF. We shall solve the extremal problem (6). Let $w_0 \in \Delta_M$ and let $D \in G$ such that $w_0 \in \Delta_M \setminus D$. Then there exists a domain $D_{w_0} = \Delta_M \setminus (Me^{i\varphi}, w_0)$ such that $g(0, z_0, D_{w_0}) \geqq g(0, z_0, D)$. In order to determine the function $\mu(w)$ we map D_{w_0} onto the upper half-plane U. The corresponding transformation is

$$W = (\zeta^2 + h^2)^{1/2}, \qquad \zeta = i(Me^{i\varphi} - w)(Me^{i\varphi} + w)^{-1}$$

where

$$W(0) = i(1 - h^2)^{1/2}, \qquad W(z_0) = i(\zeta^2(z_0) - h^2)^{1/2},$$
$$h = (M - |w_0|)(M + |w_0|)^{-1}.$$

Using the conformal invariance of Green's function, we obtain

$$g(0, z_0, D_h) = g(W(0), W(z_0), U) = \log\left|\frac{\overline{W(0)} - W(z_0)}{W(0) - W(z_0)}\right|$$

which with (7) gives us

$$\frac{|h^2 - W^2(z_0)| + 1 - h^2}{|1 - W^2(z_0)|} < \frac{1}{2}(|z_0| + |z_0|^{-1}).$$

This last inequality reduces to (8) after some simple substitutions.

If we let $M \to \infty$ in (8), then we obtain the well-known elliptic domain defined by (9) [5]. This completes the proof of Theorem 2.

REMARK. We may apply the method given by (6) and (7) to determine Koebe sets for some other subclasses of $\mathfrak{M}_M$. One can convince himself that the extremal domain for the class of convex maps consists of domains whose boundary is an arc of $\partial\Delta_M$ plus a chord of Δ_M. For the close-to-convex maps the extremal domains are those obtained from Δ_M by removing a slit along a segment emanating from w_0. Unfortunately, the corresponding formulas for the Green's functions involve transcendental functions that have made it impossible for us to find a suitable description of the Koebe sets for those classes.

However, we can establish the following result.

THEOREM 3. *The set* $\mathcal{K}(\mathfrak{M}_\infty^\alpha)$ *is given by the condition*

$$\begin{aligned} 1 + [A(1 + \cos 2\theta) + (B^2 - 1)\cos 2\theta + BC \sin 2\theta]^{1/2} \\ < [1 - (1 - D^2)^{1/2}]\,|z_0|^{-2} \end{aligned} \tag{10}$$

where

$$A = 2hd(h+d)^{-2}, \qquad B = |h-d|\,|z_0|^{-1},$$
$$D = |z_0|\,|h+d|^{-1}, \qquad C = [(1-D^2)(D^2-1+2A)]^{1/2},$$
$$h = |w|, \qquad d = |w-z_0|, \qquad \theta = \alpha - \arg z_0.$$

The set $\mathcal{K}(\mathfrak{M}_\infty^\alpha)$ is a simply connected Jordan domain if $|z_0| < (1+|\sin\theta|)^{-1/2}$, $\theta \neq 0, \pi$ and is a union of two simply connected Jordan domains which are symmetric with respect to the point $z_0/2$ if $(1+|\sin\theta|)^{-1/2} < |z_0|, \theta \neq 0, \pi$.

Proof. Let D be a domain that is convex in the direction $e^{i\alpha}$ containing the origin and z_0, and omitting a given point w_0, and let E^2 denote the open plane. There exists a ray l_α: $\arg(w-w_0)=\alpha$ such that $D\subset E^2\backslash l_\alpha$ and $g(0, z_0, D) \leqq g(0, z_0, E^2\backslash l_\alpha)$. Hence we can restrict ourselves to the domains $D_\alpha = E^2\backslash l_\alpha$ without loss in generality. Now by a translation and a rotation we can send l_α to the negative real axis and the origin and z_0 to the points $de^{i(\varphi-\alpha)}$ and $he^{i(\psi-\alpha)}$, respectively. The preceding transformation followed by the transformation $w^{1/2}$ gives us the right half-plane H. Hence we have

$$\begin{aligned} g(0, z_0, D) &= g(d^{1/2}e^{i(\varphi-\alpha)/2}, h^{1/2}e^{i(\psi-\alpha)/2}, H) \\ &= \frac{1}{2}\log\frac{d+h+2(hd)^{1/2}\cos\left(\alpha-\dfrac{\varphi+\psi}{2}\right)}{d+h-2(dh)^{1/2}\cos\frac{1}{2}(\varphi-\psi)} \qquad (11) \\ &= F(\varphi,\psi) \end{aligned}$$

where h, d, α are fixed and φ, ψ have to be chosen so that $F(\varphi, \psi)$ is a maximum. It is geometrically clear that

$$2hd\cos(\varphi-\psi) = d^2+h^2-|z_0|^2 \qquad (12)$$

and

$$h\cos\varphi - d\cos\psi = |z_0|\cos\theta \qquad (13)$$

hold. From (12) and (13), after some elementary calculations, we obtain

$$\begin{aligned} 4hd\cos^2\left(\alpha-\frac{\varphi+\psi}{2}\right) &= \frac{(h^2-d^2)\cos 2\theta - |z_0|^2(h^2+d^2)\cos 2\theta}{|z_0|^2} \\ &\quad + \frac{2hd|z_0|^2 - |h^2-d^2|\sin 2\theta}{|z_0|^2}. \end{aligned} \qquad (14)$$

Now (11), (12) and (14) give us the formula (10).

It is clear that (10) is symmetric with respect to the h and d. This means that (10) is symmetric with respect to the point $z_0/2$. However, $z_0/2 \in \mathcal{K}(\mathfrak{M}_\infty^\alpha)$ if and only if $|z_0|^2 < (1+|\sin\theta|)^{-1} = K$. Of course, $K<1$ if $\theta \neq 0$, π. Hence, if $|z_0|^2 < K$ then $\mathcal{K}(\mathfrak{M}_\infty^\alpha)$ is a simply connected Jordan domain; it is the union of two simply connected domains if $|z_0|^2 > K$. Our proof is now complete.

If $z_0 = 0$ and $\theta = \pi/2$ we obtain

$$8|w|(|w| + |\operatorname{Im} w|) < 1,$$

which defines the Koebe set for the class of univalent functions mapping Δ onto domains convex in the direction of the imaginary axis and normalized by the conditions $f(0)=0$, $f'(0)=1$.

References

1. J. Krzyż and M. O. Reade, *Koebe domains for certain classes of analytic functions*, J. Analyse Math. **18** (1967), 185–195. MR **35** #3050.

2. J. Krzyż and E. Złotkiewicz, *Koebe sets for univalent functions with two preassigned values*, Ann. Acad. Sci. Fenn. Ser. AI (to appear).

3. Z. Lewandowski, *Sur certaines classes de fonctions univalentes dans le cercle-unité*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **13** (1959), 115–126. MR **22** #11137.

4. G. Pick, *Über die konforme Abbildung eines kreises* · · · , Wien. Berichte **126** (1917), 247–263.

5. W. Rogosinski, *Über den Wertevorrat einer analytischen Funktion, von der zwei Werte vorgegeben sind*, Compositio Math. **3** (1936), 199–226.

University of Michigan, Ann Arbor, Michigan 48104

Mariae Curie-Skłodowska University, Lublin, Poland