ON UNIVALENT FUNCTIONS WITH TWO
PREASSIGNED VALUES

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Abstract. Let \( \mathcal{M}_M \) denote the class of functions analytic and
univalent in the unit disc \( \Delta \) subject to the conditions

\[
f(0) = 0, \quad f(z_0) = z_0, \quad |f(z)| < M,
\]

where \( z_0 \) is a fixed point of \( \Delta \) and \( 1 \leq M \leq \infty \).

In the present note, we determine by the method of circular
symmetrization, the exact value of the "Koebe constant" for the
class \( \mathcal{M}_M \). We also determine Koebe sets for the class \( \mathcal{M}_M^* \), consisting
of the starlike functions, and for \( \mathcal{M}_M^* \), consisting of all functions
mapping \( \Delta \) onto domains convex in the direction \( e^{i\alpha} \).

By "Koebe set" we understand the set \( \mathcal{K}(\mathcal{M}_M), \mathcal{K}(\mathcal{M}_M^*) \)

= \( \bigcap_{f \in \mathcal{M}_M} f(\Delta) \).

1. Introduction. Let \( \mathcal{M}_M \) denote the class of functions analytic and
univalent in the unit disc \( \Delta \) subject to the conditions

\[
(1) \quad f(0) = 0, \quad f(z_0) = z_0, \quad |f(z)| < M
\]

where \( z_0 \) is a fixed point of \( \Delta \) and \( 1 < M < \infty \).

So far as we know there are no results concerning class \( \mathcal{M}_M \),
whereas many authors have considered certain extremal problems in
the case \( M = \infty \).

In the present note, we determine by the method of the circular
symmetrization, the exact value of the Koebe constant for the class
\( \mathcal{M}_M \). We also determine Koebe sets for the class \( \mathcal{M}_M^* \), consisting of the
starlike functions, and for \( \mathcal{M}_M^* \), consisting of all functions mapping \( \Delta \)
on to domains convex in the direction \( e^{i\alpha} \).

By "Koebe set" [1] we understand the set \( \mathcal{K}(\mathcal{M}_M), \mathcal{K}(\mathcal{M}_M^*) \)

= \( \bigcap_{f \in \mathcal{M}_M} f(\Delta) \).

2. Main results. 2.1. We start with the determination of the
Koebe constant for the class \( \mathcal{M}_M \).

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Theorem 1. The image-domain of \( \Delta \) under each function of the class \( \mathcal{M}_M \) contains the disc \( |w| < r(M) \) where

\[
r(M) = 2\delta^2 - M - 2\delta(\delta^2 - M)^{1/2},
\]
\[
\delta = (M - |z_0|)(1 - |z_0|)^{-1}.
\]

The number \( r(M) \) cannot be replaced by any greater number without additional assumptions on the functions of the class \( \mathcal{M}_M \).

Proof. Suppose that \( \Omega = \Omega(\Delta) \) and \( \rho(a, b, \Omega) \) denotes the hyperbolic distance of the points \( a, b \) with respect to the domain \( \Omega \).

Since the hyperbolic distance is a conformal invariant, we have

\[
\text{arc th } |z_0| = \rho(0, z_0, f(\Delta)), \quad f \in \mathcal{M}_M.
\]

Let \( \Omega^* \) be a domain obtained from \( \Omega \) by using circular symmetrization with respect to the ray \([0, z_0, \infty] \). It is clear that the origin and \( z_0 \) are in \( \Omega^* \). Moreover, it is well known that circular symmetrization decreases the hyperbolic distance, so that

\[
\rho(0, z_0, \Omega^*) \leq \rho(0, z_0, \Omega^*)
\]

holds. However, each domain \( \Omega^* \) is contained in a domain \( D_h \), 
\[D_h = \Delta_M \backslash \{ -M e^{i\alpha}, -he^{i\alpha} \} \] for some \( h, h > 0 \), where \( \Delta_M \) is the disc \( |w| < M \). Hence

\[
\rho(0, z_0, \Omega^*) \leq \rho(0, z_0, D_h).
\]

The conditions (2), (3) and (4) now yield our basic inequality

\[
\text{arc th } |z_0| \leq \rho(0, z_0, D_h).
\]

For our purpose it is sufficient to consider \( z_0 = r \). Now let \( \varphi(z) \), \( \varphi(0) = 0 \) be the function that maps the domain \( D_h \), which is now \( \Delta_M \backslash \{ -M, -h \} \) conformally onto the unit disc. Then we have

\[
\rho(0, z_0, D_h) = \text{arc th } |\varphi(r)|,
\]

where

\[
\varphi(r) = \frac{2Mr}{d(r^2 + M^2) + 2Mr(1 - d) + ([d(M - r)^2 + 2Mr]^2 - 4M^2r^2)^{1/2}},
\]
\[
d = 4hM(M + h)^{-2}.
\]

Thus (5) yields the inequality

\[
d(M - r)^2 + 2Mr + ([d(M - r)^2 + 2Mr]^2 - 4M^2r^2)^{1/2} \geq 2M,
\]

which holds if
or
\[ d \geq M(1 - r)^2(M - r)^{-2} \]

The sign of equality in these last considerations occurs for the function \( f(z) \) given by the equation
\[ \frac{f(z)}{[M - f(z)]^2} = \left( \frac{1 - r}{M - r} \right)^2 \frac{z}{(1 - z)^2} . \]

This completes our proof.

**Corollary 1.** For the cases \( M = \infty \) and \( z_0 = 0 \) we have
\[ r(\infty) = \frac{1}{2}(1 - |z_0|^2), \quad r_0(M) = 2M^2 - M - 2M(M^2 - M)^{1/2}, \]
respectively. Both constants are well known ([3], [4], respectively).

2.2. Now we shall be concerned with the determination of Koebe sets for starlike functions of the class \( \mathfrak{W}_M \). Let \( \Delta_M \) be the disc \( \{ w : |w| < M \} \) and let \( G \) be the family of all its subdomains that are (i) starshaped with respect to the origin and (ii) contain the fixed point \( z_0 \). For \( D \in G \), let \( g(w, z_0, D) \) be the Green's function with pole at \( z_0 \), and let \( \mu(w_0) \) be defined as a solution of the following extremal problem:
\[ \mu(w_0) = \max_{D \in G} \min_{w_0 \in \Delta_M \setminus D} g(0, z_0, D). \]

The following formula for \( \mathcal{K}(\mathfrak{W}_M) \) was obtained by Krzyż and Zlotkiewicz [2]:
\[ \mathcal{K}(\mathfrak{W}_M) = \{ w : \mu(w) < -\log |z_0| \}. \]
Hence, in order to determine \( \mathcal{K}(\mathfrak{W}_M) \) it is sufficient to solve the extremal problem (6). We shall use (7) for the two classes \( \mathfrak{W}_M, \mathfrak{W}_M^* \) consisting of all starlike functions of the class \( \mathfrak{W}_M \) and of those functions mapping \( \Delta \) onto domains convex in the given direction \( e^{i\alpha} \), respectively.

**Theorem 2.** The Koebe set of the class \( \mathfrak{W}_M^* \) is determined by the condition
\[ \left| w - z_0 \right| \left( \frac{M - wz_0}{w} \right)^2 + \left( \frac{Mw + z_0}{w} \right)^2 \frac{1}{|w|} < \frac{1}{2}(1 + |z_0|^2). \]

The limit case, \( M = \infty \), is
We shall solve the extremal problem (6). Let $w_0 \in \Delta_M$ and let $D \in G$ such that $w_0 \in \Delta_M \setminus D$. Then there exists a domain $D_w = \Delta_M \setminus (M e^{i\varphi}, w_0)$ such that $g(0, z_0, D_w) \geq g(0, z_0, D)$. In order to determine the function $\mu(w)$ we map $D_w$ onto the upper half-plane $U$. The corresponding transformation is

$$W = (\xi^2 + h^2)^{1/2}, \quad \xi = i(M e^{i\varphi} - w)(M e^{i\varphi} + w)^{-1}$$

where

$$W(0) = i(1 - h^2)^{1/2}, \quad W(z_0) = i(\xi(z_0) - h^2)^{1/2},$$

$$h = (M - |w_0|)(M + |w_0|)^{-1}.$$

Using the conformal invariance of Green's function, we obtain

$$g(0, z_0, D_w) = g(W(0), W(z_0), U) = \log \left| \frac{W(0) - W(z_0)}{W(0) - W(z_0)} \right|$$

which with (7) gives us

$$\frac{h^2 - W^2(z_0)}{1 - W^2(z_0)} + \frac{1 - h^2}{2} (|z_0| + |z_0|^{-1}).$$

This last inequality reduces to (8) after some simple substitutions.

If we let $M \to \infty$ in (8), then we obtain the well-known elliptic domain defined by (9) [5]. This completes the proof of Theorem 2.

Remark. We may apply the method given by (6) and (7) to determine Koebe sets for some other subclasses of $\mathfrak{M}$. One can convince himself that the extremal domain for the class of convex maps consists of domains whose boundary is an arc of $\partial \Delta_M$ plus a chord of $\Delta_M$. For the close-to-convex maps the extremal domains are those obtained from $\Delta_M$ by removing a slit along a segment emanating from $w_0$. Unfortunately, the corresponding formulas for the Green's functions involve transcendental functions that have made it impossible for us to find a suitable description of the Koebe sets for those classes.

However, we can establish the following result.

Theorem 3. The set $\mathcal{K}(\mathfrak{M}_c)$ is given by the condition

$$1 + \left[ A(1 + \cos 2\theta) + (B^2 - 1) \cos 2\theta + BC \sin 2\theta \right]^{1/2} < [1 - (1 - D^2)^{1/2}] |z_0|^{-2}$$

where
The set $\mathcal{K}(\mathfrak{M}_0^\theta)$ is a simply connected Jordan domain if $|z_0| < (1 + |\sin \theta|)^{-1/2}$, $\theta \neq 0, \pi$ and is a union of two simply connected Jordan domains which are symmetric with respect to the point $z_0/2$ if $(1 + |\sin \theta|)^{-1/2} < |z_0|$, $\theta \neq 0, \pi$.

Proof. Let $D$ be a domain that is convex in the direction $e^{ia}$ containing the origin and $z_0$, and omitting a given point $w_0$, and let $E^2$ denote the open plane. There exists a ray $l_\alpha$: arg $(w-w_0) = \alpha$ such that $D \subseteq E^2 \setminus l_\alpha$ and $g(0, z_0, D) \leq g(0, z_0, E^2 \setminus l_\alpha)$. Hence we can restrict ourselves to the domains $D_\alpha = E^2 \setminus l_\alpha$ without loss in generality. Now by a translation and a rotation we can send $l_\alpha$ to the negative real axis and the origin and $z_0$ to the points $d e^{i(\theta-\alpha)}$ and $h e^{i(\theta-\alpha)}$, respectively. The preceding transformation followed by the transformation $w^{1/2}$ gives us the right half-plane $H$. Hence we have

$$g(0, z_0, D) = g(d^{1/2} e^{i(\theta-\alpha)/2}, h^{1/2} e^{i(\theta-\alpha)/2}, H)$$

$$= \frac{1}{2} \log \frac{d + h + 2(hd)^{1/2} \cos \left(\alpha - \frac{\varphi + \psi}{2}\right)}{d + h - 2(dh)^{1/2} \cos \frac{1}{2}(\varphi - \psi)}$$

$$= F(\varphi, \psi)$$

where $h, d, \alpha$ are fixed and $\varphi, \psi$ have to be chosen so that $F(\varphi, \psi)$ is a maximum. It is geometrically clear that

$$2hd \cos (\varphi - \psi) = d^2 + h^2 - |z_0|^2$$

and

$$h \cos \varphi - d \cos \psi = |z_0| \cos \theta$$

hold. From (12) and (13), after some elementary calculations, we obtain

$$4hd \cos^2 \left(\alpha - \frac{\varphi + \psi}{2}\right) = \frac{(h^2 - d^2) \cos 2\theta - |z_0|^2(h^2 + d^2) \cos 2\theta}{|z_0|^2}$$

$$+ \frac{2hd |z_0|^2 - |h^2 - d^2| \sin 2\theta}{|z_0|^2}.$$ 

Now (11), (12) and (14) give us the formula (10).
It is clear that (10) is symmetric with respect to the $h$ and $d$. This means that (10) is symmetric with respect to the point $z_0/2$. However, $z_0/2 \in \mathcal{K}(\mathbb{D}_2)$ if and only if $|z_0|^2 < (1 + |\sin \theta|)^{-1} = K$. Of course, $K < 1$ if $\theta \neq 0, \pi$. Hence, if $|z_0|^2 < K$ then $\mathcal{K}(\mathbb{D}_2)$ is a simply connected Jordan domain; it is the union of two simply connected domains if $|z_0|^2 > K$. Our proof is now complete.

If $z_0 = 0$ and $\theta = \pi/2$ we obtain
\[ 8 |w| (|w| + |\text{Im} w|) < 1, \]
which defines the Koebe set for the class of univalent functions mapping $\Delta$ onto domains convex in the direction of the imaginary axis and normalized by the conditions $f(0) = 0, f'(0) = 1$.

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