ON THE COEFFICIENTS OF BAZILEVIĆ FUNCTIONS

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Abstract. Let \( B(\alpha) \) denote the class of normalized \((f(0) = 0, f'(0) = 1)\) Bazilević functions of type \( \alpha \) defined in \( \Delta : |z| < 1 \), i.e. \( f(z) = \left[ \alpha \beta P(\zeta) g(\zeta) \zeta^{1-1-i\beta} \right]^{1/\alpha} \) where \( g(z) \) is starlike in \( \Delta \), \( P(\zeta) \) is regular with \( \Re P(\zeta) > 0 \) in \( \Delta \) and \( \alpha > 0 \). Let \( B_m(\alpha) \) denote the subclass of \( B(\alpha) \) which is \( m \)-fold symmetric \((f(e^{2\pi i m z}) = e^{2\pi i m f(z)}, m = 1, 2, \cdots)\).

Functions in \( B(\alpha) \) have been shown to be univalent. The authors obtain sharp coefficient inequalities for functions in \( B_m(1/N) \) where \( N \) is a positive integer. In addition an example of a Bazilević function which is not close-to-convex is given.

Let \( S \) denote the class of functions

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

that are univalent in the unit disk \( \Delta : |z| < 1 \). If \( g(z) \) is starlike in \( \Delta \), \( P(z) \) is regular with \( \Re P(z) > 0 \) in \( \Delta \) and \( \alpha > 0 \), then the function

\[
f(z) = \left[ \alpha \int_{0}^{z} P(\xi) g(\xi) \xi^{1-\xi-i\beta} d\xi \right]^{1/\alpha}
\]

has been shown by Bazilević [1] (see also Pommerenke [6]) to be a regular and univalent function in \( \Delta \). The powers appearing in the formula are meant as principal values. We shall call a function in \( S \) satisfying (1) a Bazilević function of type \( \alpha \) and denote the class of such functions by \( B(\alpha) \). Note that \( B(1) \) is the class of normalized close-to-convex functions.

A function \( f(z) \) analytic in \( \Delta \) is said to be \( m \)-fold symmetric \((m = 2, 3, \cdots)\) if \( f(e^{2\pi i m z}) = e^{2\pi i m f(z)} \). In particular, every odd \( f(z) \) is 2-fold symmetric. Let \( S_m \) denote the subclass of \( S \) consisting of those \( f(z) \) that are \( m \)-fold symmetric. We similarly define \( B_m(\alpha) \). A simple argument shows that \( f \in S_m \) is characterized by having a power series of the form \( f(z) = z + a_{m+1} z^{m+1} + a_{2m+1} z^{2m+1} + \cdots \).

The Bieberbach conjecture remains unsettled for functions in \( B(\alpha) \) except for the case \( \alpha = 1/N \), where \( N \) is a positive integer (Zamorski

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C. Pommerenke [5] has obtained sharp coefficient inequalities for functions in $B_m(1)$. In this paper we shall be concerned mainly with obtaining sharp coefficient inequalities for functions in $B_m(1/N)$. The method is different from the methods of Zamorski and Pommerenke and our results include theirs.

Many of the properties of Bazilevič functions of type $\alpha$, $0 < \alpha < 1$, coincide with properties of close-to-convex functions, including some of the results of this paper. We will show, at the end of the paper, an example of a Bazilevič function which is not close-to-convex.

The notation $g(z) \ll h(z)$ ("$g(z)$ is majorized by $f(z)$") will mean that if $g(z) = \sum_0^\infty b_n z^n$ and $h(z) = \sum_0^\infty c_n z^n$ then $|b_n| \leq c_n$ for $n = 0, 1, \cdots$ (see, for example, [3, p. 69]).

We shall need the following lemmas, the first of which is well known.

**Lemma 1.** (a) $\psi \in S_m$ if and only if $\psi(z) = \left[ f(z^m) \right]^{1/m}$ where $f \in S$. (b) $\psi(z)$ is $m$-fold symmetric and starlike if and only if $\psi(z) = \left[ f(z^m) \right]^{1/m}$ where $f(z)$ is starlike.

**Lemma 2.** If $f(z)$ is $m$-fold symmetric and starlike and $\gamma > 0$ then

$$[f(z)/z]^{\gamma} \ll \left| f'(0) \right|^{\gamma(1 - z^m)^{-2\gamma/m}}.$$  

**Proof.** A function $f(z)$ which is $m$-fold symmetric and starlike has a Herglotz representation given by

$$\log \frac{f(z)}{zf'(0)} = \int_0^{2\pi} \log \frac{1}{(1 - z^m e^{-i\phi})^{2/m}}d\mu(\phi),$$

where $\mu(\phi)$ is nondecreasing on $[0, 2\pi]$ and $\mu(2\pi) - \mu(0) = 1$. Thus

$$\log \frac{f(z)}{zf'(0)} = \frac{2}{m} \int_0^{2\pi} \sum_{n=1}^\infty \frac{z^m e^{-in\phi}}{n} d\mu(\phi)$$

$$= \frac{2}{m} \sum_{n=1}^\infty \left[ \int_0^{2\pi} e^{-in\phi} d\mu(\phi) \right] \frac{z^m}{n} \ll \frac{2}{m} \sum_{n=1}^\infty \frac{z^m}{n},$$

and consequently

$$\log(f(z)/zf'(0)) \ll \log(1 - z^m)^{-2/m}.$$  

If $\gamma > 0$, then

$$\gamma \log(f(z)/zf'(0)) \ll \gamma \log(1 - z^m)^{-2\gamma/m},$$

or

$$\log[\left| f(z)/zf'(0) \right|^\gamma] \ll \log(1 - z^m)^{-2\gamma/m}.$$
Since exponentiation preserves the majorization property we obtain
\[ f(z)/zf'(0) \approx (1 - z^m)^{-2/m}, \] which is equivalent to the desired result.

**Theorem 1.** \( \phi(z) \in B_m(\alpha) \) if and only if
\[
\phi(z) = [f(z^m)]^{1/m}
\]
where \( f(z) \in B_1(\alpha/m) \).

**Proof.** (i) If \( \phi(z) \in B_m(\alpha) \), then on differentiating (1) we obtain
\[
z\phi'(z)\phi(z)^{-1} = g(z)^aP(z),
\]
where \( g \) is starlike and \( \text{Re } P(z) > 0 \). By Lemma 1 (a) there exists an \( f \in S \) satisfying (2), and substitution of (2) in (3) yields
\[
z^m f'(z^m)f(z^m)^{a/m-1} = g(z)^aP(z).
\]
Replacing \( z \) by \( e^{2k \pi i/m}z \) we have
\[
z^m f'(z^m)f(z^m)^{a/m-1} = g(e^{2k \pi i/m}z)^aP(e^{2k \pi i/m}z),
\]
for \( k = 0, 1, \ldots, m-1 \). If we multiply the \( m \) equations in (4) and then take the \( m \)th root we obtain
\[
z^m f'(z^m)f(z^m)^{a/m-1} = \left[ \prod_{k=0}^{m-1} g(e^{2k \pi i/m}z)^aP(e^{2k \pi i/m}z) \right]^{1/m}.
\]
It is easily verified that the function \( \left[ \prod_{k=0}^{m-1} g(e^{2k \pi i/m}z) \right]^{1/m} \) is \( m \)-fold symmetric and starlike and hence, by Lemma 1 (b), can be written as \( h(z^m)^{1/m} \) where \( h \) is starlike. It is also easily verified that
\[
\prod_{k=0}^{m-1} P(e^{2k \pi i/m}z)^{1/m} = c_0 + c_m z^m + \cdots = Q(z^m),
\]
say, where \( \text{Re } Q(z) > 0 \). Thus we have
\[
z^m f'(z^m)f(z^m)^{a/m-1} = h(z^m)^{a/m}Q(z^m),
\]
or \( f \in B(\alpha/m) \).

(ii) Conversely, if \( f \in B(\alpha/m) \) we have
\[
zf'(z)f(z)^{a/m-1} = h(z)^{a/m}Q(z),
\]
where \( h \) is starlike and \( \text{Re } Q(z) > 0 \). Thus
\[
z^m f'(z^m)f(z^m)^{a/m-1} = h(z^m)^{a/m}Q(z^m)
\]
and, if we let \( \phi(z) = [f(z^m)]^{1/m} \), we obtain
\[
z\phi'(z)\phi(z)^{-1} = [h(z^m)^{1/m}]^{a}Q(z^m).
\]
By Lemma 1 (b), \( g(z) = [h(z^m)]^{1/m} \) is \( m \)-fold symmetric starlike, and if we write \( P(z) = Q(z^m) \) we have
\[
z^k f'(z) f(z)^{m-1} = g(z)^m P(z).
\]
Thus \( f(z) \in B_m(\alpha) \), and this completes the proof of the theorem.

**Theorem 2.** If \( f(z) \in B_m(\alpha) \), then \( \left[ f(z)/z \right]^\alpha \ll (1 - z^m)^{-2\alpha/m} \).

**Proof.** The function
\[
F(z) = \left[ f(z)/z \right]^\alpha = 1 + A_1 z^m + A_2 z^{2m} + \cdots
\]
satisfies the differential equation
\[
z F'(z) + \alpha F(z) = \alpha f'(z) f(z)^{\alpha-1}/z^{\alpha-1},
\]
or, using (5),
\[
z F'(z) + \alpha F(z) = \alpha \left[ [h(z^m)]^{1/m}/z \right]^\alpha Q(z^m),
\]
where \( [h(z^m)]^{1/m} \) is \( m \)-fold symmetric and starlike and \( \Re Q(z^m) > 0 \).

Hence, by Lemma 2, we have
\[
[ [h(z^m)]^{1/m}/z \right]^\alpha \ll \left| h'(0) \right|^{\alpha/m} (1 - z^m)^{-2\alpha/m}.
\]
Also, if \( Q(z^m) = c_0 + c_m z^m + c_2 z^{2m} + \cdots \) then
\[
Q(z^m) \ll \left| c_0 \right| \frac{1 + z^m}{1 - z^m}
\]
[4, p. 170]. Since multiplication preserves the majorization property, (6), (7) and (8) yield
\[
z F'(z) + \alpha F(z) \ll \alpha \left| h'(0) \right|^{\alpha/m} \left| c_0 \right| \frac{1 + z^m}{(1 - z^m)^{1+2\alpha/m}}.
\]
Setting \( z = 0 \) in (6) we obtain \( \alpha = \frac{\alpha [h'(0)]^{\alpha/m} c_0}{} \), and consequently
\[
z F'(z) + \alpha F(z) \ll \alpha \frac{1 + z^m}{(1 - z^m)^{1+2\alpha/m}}.
\]

On comparing coefficients we have
\[
| (mn + \alpha) A_n | \leq \alpha \left[ \begin{pmatrix} 2\alpha/m + n \\ n \end{pmatrix} + \begin{pmatrix} 2\alpha/m + n - 1 \\ n - 1 \end{pmatrix} \right],
\]
\[
| A_n | \leq \begin{pmatrix} 2\alpha/m + n - 1 \\ n \end{pmatrix},
\]
or equivalently, \( \left[ f(z)/z \right]^\alpha \ll (1 - z^m)^{-2\alpha/m} \).
Corollary. If \( \phi \in B_m(1/N) \), where \( N \) is a positive integer and 
\[ \phi(z) = z + a_{m+1}z^{m+1} + \cdots \]
then
\[
(9) \quad |a_{mn+1}| \leq \left( \frac{2/m + n - 1}{n} \right).
\]
In particular for \( \phi \in B(1/N) \) we have \( |a_n| \leq n \) and for \( \phi \in B_2(1/N) \) we have \( |a_{2n+1}| \leq 1 \).

Proof. With \( \alpha = 1/N \) we have
\[
|\phi(z)/z|^{1/N} \ll \left( 1 - z^n \right)^{-2/m}.
\]
Multiplying this result by itself \( N \) times we obtain \( \phi(z)/z \ll \left( 1 - z^n \right)^{-2/m} \), which is equivalent to the desired result.

The inequality (9) is sharp, as can be seen by considering the function 
\[
z(1-z^n)^{-2/m} \text{ which is in } B_m(1/N).
\]
We conclude by constructing an example of a function \( f \in B(1/2) \) such that \( f \) is not close-to-convex. With an appropriate \( c > 0 \), let 
\( w = \phi(z) = z + \cdots \) be the odd close-to-convex function that maps \( \Delta \) onto the \( w \)-plane slit along the half-lines \( \Re w \geq 0, \Im w = c \) and \( \Re w \leq 0, \Im w = -c \). Since \( \phi \in B_2(1) \), by Theorem 1 we have \( \phi(z) = [f(z^2)]^{1/2} \), where \( f(z) \in B(1/2) \). But the transformation \( \zeta = \xi + in = f(z) \) maps \( \Delta \) onto the \( \zeta \)-plane slit along the portion of the parabola 
\( \xi = (\eta/2c)^2 - c^2 \) defined for \( \eta \geq 0 \), and this slit clearly cannot be expressed as a union of half-lines. It follows by a well-known geometric criterion (see, for example, Bielecki and Lewandowski [2, p. 61]) that the domain is not close-to-convex.

References


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