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AN INTEGRAL REPRESENTATION FOR GENERALIZED TEMPERATURES IN TWO SPACE VARIABLES

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ABSTRACT. An integral representation is derived for a function which satisfies the generalized heat equation in one of the space variables and the adjoint generalized heat equation in the other space variable.

In recent papers [2]–[7], series and integral representation theories were developed for generalized temperature functions. In this note, we derive an integral representation for a function which satisfies the generalized heat equation in one of the space variables and the adjoint generalized heat equation in the other space variable.

A generalized temperature function is a C^2 solution $u(x, t)$ of the generalized heat equation

$$\Delta_x(u(x, t)) = (\partial/\partial t)u(x, t),$$

where $\Delta_x f(x) = f''(x) + (2\nu/x)f'(x)$, $\nu > 0$. The adjoint generalized heat equation is given by

$$\Delta_x u(x, t) + (\partial/\partial t)u(x, t) = 0.$$

The fundamental solution of the generalized heat equation is the function

$$G(x; t) = \int_0^\infty e^{-tu^2}\mathcal{J}(xu)d\mu(u) = \left(\frac{1}{2t}\right)^{\nu+1/2} e^{-x^2/4t},$$

where

$$\mathcal{J}(z) = 2^{\nu-1/2}\Gamma(\nu + \tfrac{1}{2})z^{1/2-\nu}J_{\nu-1/2}(z),$$

$$d\mu(z) = \frac{1}{2^{\nu-1/2}\Gamma(\nu + \frac{1}{2})} z^{2\nu}dz,$$

$J_\alpha(z)$ being the ordinary Bessel function of order α. If

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$$D(x, y, z) = \frac{2^{2\nu-5/2}[\Gamma(\nu+\frac{1}{2})]^2}{\Gamma(\nu)\pi^{1/2}}(xyz)^{1-2\nu}[\Delta(x, y, z)]^{2\nu-2},$$

where $\Delta(x, y, z)$ is the area of a triangle whose sides are x, y, z if there is such a triangle, and otherwise, $D(x, y, z)=0$, the associated function $f(x, y)$ corresponding to a function $f(x)$ is given by

$$f(x, y) = \int_0^\infty f(u)D(x, y, u)d\mu(u), \qquad 0 < x, y < \infty.$$

The function associated with the fundamental solution $G(x; t)$ is

$$G(x, y; t) = \int_0^\infty e^{-tu^2}\mathcal{J}(xu)\mathcal{J}(yu)d\mu(u) = \left(\frac{1}{2t}\right)^{\nu+1/2} e^{-(x^2+y^2)/4t}\mathcal{J}\left(\frac{xy}{2t}\right),$$

where

$$\mathcal{J}(z) = 2^{\nu-1/2}\Gamma(\nu+\tfrac{1}{2})z^{1/2-\nu}I_{\nu-1/2}(z),$$

$I_\alpha(z)$ being the Bessel function of imaginary argument and order α. A bounded function $f(x)$ for which

$$\sum_{j=1}^n \sum_{k=1}^n a_j\bar{a}_k f(x_j, x_k) \geqq 0$$

for any $x_1, x_2, \cdots, x_n > 0$ and arbitrary complex numbers $a_1, a_2, \cdots, a_n$ is said to be positive definite.

We establish two theorems, each a consequence of two basic lemmas which give integral representations for functions of one space variable.

LEMMA 1. *A function $u(x, t)$ has the representation*

$$u(x, t) = \int_0^\infty e^{-tu^2}\mathcal{J}(xu)d\alpha(u), \qquad t > 0,$$

where $\alpha(u)$ is a nondecreasing function iff

(i) $\Delta_x u(x, t) = -(\partial/\partial t)u(x, t), t > 0,$

(ii) $u(x, t) \geqq 0, t > 0.$

PROOF. The necessity of the conditions is immediate on noting that $\mathcal{J}(xu) \geqq 0$ and that $\Delta_x\mathcal{J}(xu) = u^2\mathcal{J}(xu)$, with differentiation under the integral sign justifiable.

To establish sufficiency, consider the function

$$v(x, t) = G(x; t)u\left(\frac{x}{t}, \frac{1}{t}\right), \qquad t > 0.$$

Then it readily follows that $v(x, t)$ is a nonnegative generalized temperature function for $t>0$. Hence by [2, p. 49], we have

$$v(x, t) = \int_0^\infty G(x, y; t)d\beta(y),$$

with $\beta(y)$ nondecreasing, or

$$u(x, t) = \int_0^\infty e^{-y^2 t}\mathcal{I}(xy)d\alpha(y)$$

where $\alpha(y) = \beta(2y)$, and the proof is complete.

LEMMA 2. *A function $u(x, t)$ has the representation*

$$u(x, t) = \int_0^\infty e^{-tu^2}\mathcal{J}(xu)d\alpha(u), \qquad t > 0,$$

where $\alpha(u)$ is nondecreasing iff

(i) $\Delta_x u(x, t) = (\partial/\partial t)u(x, t)$, $t>0$,

(ii) *$u(z, t)$ is analytic for each $t>0$ and* $|\operatorname{Re} z| < R$,

(iii) $u(ix, t) \geqq 0$, $t>0$.

PROOF. The necessity of the conditions is immediate with the analyticity of $u(z, t)$ a consequence of Theorem 5.3 of [2].

Conversely, since $u(iy, t) \geqq 0$ and $u(iy, t)$ is a solution of the adjoint generalized heat equation for $t>0$, the result follows by the preceding lemma.

Combining these two results, we have the following.

THEOREM 3. *A function $u(x, y, t)$ has the representation*

$$u(x, y, t) = \int_0^\infty e^{-tu^2}\mathcal{J}(xu)\mathcal{I}(yu)d\mu(u), \qquad t > 0,$$

with $\alpha(u)$ nondecreasing iff

(i) $\Delta_x u(x, y, t) = -\Delta_y(x, y, t) = (\partial/\partial t)u(x, y, t)$,

(ii) *for $y \geqq 0$, $t>0$, $u(z, y, t)$ is analytic for* $|\operatorname{Re} z| < R$,

(iii) *for each $x \geqq 0$, $y \geqq 0$, $t>0$, $u(ix, y, t) \geqq 0$.*

PROOF. The necessity of the conditions are readily verified.

Conversely, an appeal to Lemma 1, yields, for fixed x,

$$u(x, y, t) = \int_0^\infty e^{-tu^2}\mathcal{I}(yu)d\varphi(x, u),$$

where, for each x, $\varphi(x, u)$ is nondecreasing. Hence

$$u(x, 0, t) = \int_0^\infty e^{-tu^2} d\varphi(x, u).$$

But, $u(x, 0, t)$ satisfies the conditions of Lemma 2 so that

$$u(x, 0, t) = \int_0^\infty e^{-tu^2} \mathcal{J}(xu) d\alpha(u),$$

with $\alpha(u)$ nondecreasing. Consequently,

$$\int_0^\infty e^{-tu^2} [\mathcal{J}(xu) d\alpha(u) - d\varphi(x, u)] = 0,$$

and since the left-hand side is a Laplace transform, it follows by uniqueness, that, for each fixed x,

$$\mathcal{J}(xu) d\alpha(u) = d\varphi(x, u),$$

yielding the desired representation for $u(x, y, t)$.

An example illustrating the theorem is given by

$$\left(\frac{1}{2t}\right)^{\nu+1/2} e^{(-x^2+y^2)/4t} \mathcal{J}\left(\frac{xy}{2t}\right) = \int_0^\infty e^{-tu^2} \mathcal{J}(xu) \mathcal{J}(yu) d\mu(u).$$

We establish criteria for a similar representation, but with $\alpha(u)$ a nondecreasing bounded function, again, by first proving two basic lemmas.

LEMMA 4. *A necessary and sufficient condition that*

$$u(x, t) = \int_0^\infty e^{-tu^2} \mathcal{J}(xu) d\alpha(u), \qquad t > 0,$$

with $\alpha(u)$ nondecreasing and bounded, is that

(i) $\Delta_x u(x, t) = (\partial/\partial t) u(x, t)$, $t > 0$,

(ii) $u(x, t) > -M$ *for some* $M > 0$, $t > 0$,

(iii) $u(x, 0+)$ *exists and is positive definite.*

PROOF. If the integral representation holds, then (i) is immediate; (ii) follows from the fact that

$$|u(x, t)| \leqq \int_0^\infty d\alpha(u) < \infty, \qquad t > 0;$$

and (iii) from [1].

Conversely, we note that by (i) and (ii), $u(x, t) + M$ is a positive generalized temperature function, and hence, by Corollary 8.6 of [2],

$$u(x, t) = \int_0^\infty G(x, y; t)u(y, 0+)d\mu(y), \qquad t > 0.$$

But by [1], (iii) implies that

$$u(x, 0+) = \int_0^\infty \mathcal{J}(xu)d\alpha(u),$$

with $\alpha(u)$ nondecreasing and bounded. Hence

$$\begin{aligned} u(x, t) &= \int_0^\infty G(x, y; t)d\mu(y) \int_0^\infty \mathcal{J}(yu)d\alpha(u) \\ &= \int_0^\infty d\alpha(u) \int_0^\infty \mathcal{J}(yu)G(x, y; t)d\mu(y) \\ &= \int_0^\infty e^{-tu^2}\mathcal{J}(ux)d\alpha(u), \end{aligned}$$

the interchange in order of integration being justifiable by Fubini's theorem, since $\alpha(u)$ is a nondecreasing bounded function. Thus the proof is complete.

LEMMA 5. *A necessary and sufficient condition that*

$$u(x, t) = \int_0^\infty e^{-tu^2}\mathcal{J}(xu)d\alpha(u), \qquad t > 0,$$

with $\alpha(u)$ nondecreasing and bounded is that

(i) $\Delta_x u(x, t) = -(\partial/\partial t)u(x, t)$, $t>0$,

(ii) *$u(z, t)$ is analytic for each $t>0$ and $|\operatorname{Re} z| < R$,*

(iii) *$u(ix, t) > -M$ for some $M>0$, $t>0$,*

(iv) *$u(ix, 0+)$ exists and is positive definite.*

PROOF. The necessity of the conditions are immediate and the sufficiency follows on noting that $u(x, -t)$ is a generalized temperature for $t<0$, and hence $u(ix, t)$ is a generalized temperature for $t>0$. Thus $u(ix, t)$ satisfies the conditions of Lemma 4 and hence the desired integral representation follows for $u(x, t)$.

Lemmas 4 and 5 yield the following result whose proof is established analogously to that for Theorem 3 and hence will be omitted.

THEOREM 6. *A function $u(x, y, t)$ has the representation*

$$u(x, y, t) = \int_0^\infty e^{-tu^2}\mathcal{J}(xu)\mathcal{J}(yu)d\alpha(u), \qquad t > 0,$$

with $\alpha(u)$ nondecreasing and bounded iff

(i) $\Delta_x u(x, y, t) = -\Delta_y u(x, y, t) = (\partial/\partial t)u(x, y, t)$, $t>0$.
(ii) *For* $x \geqq 0$, $t>0$, $u(x, z, t)$ *is analytic for* $|\operatorname{Re} z| < R$,
(iii) $u(x, iy, t) > -M$,
(iv) $u(x, 0, 0+)$, $u(0, iy, 0+)$ *exist and are positive definite.*

The theorem is illustrated by the function

$$u(x, y, t) = e^{-ta^2}\mathcal{J}(xa)\mathcal{J}(ya)$$

which satisfies the conditions of the theorem and which has the representation

$$u(x, y, t) = \int_0^\infty e^{-tu^2}\mathcal{J}(xu)\mathcal{J}(yu)d\alpha(u),$$

where $\alpha(u)$ is constant except for a unit positive jump at $u=a$.

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