AN INTEGRAL REPRESENTATION FOR GENERALIZED TEMPERATURES IN TWO SPACE VARIABLES

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Abstract. An integral representation is derived for a function which satisfies the generalized heat equation in one of the space variables and the adjoint generalized heat equation in the other space variable.

In recent papers [2]–[7], series and integral representation theories were developed for generalized temperature functions. In this note, we derive an integral representation for a function which satisfies the generalized heat equation in one of the space variables and the adjoint generalized heat equation in the other space variable.

A generalized temperature function is a $C^2$ solution $u(x, t)$ of the generalized heat equation

$$\Delta_x u(x, t) = (\partial/\partial t)u(x, t),$$

where $\Delta_x f(x) = f''(x) + (2\nu/x)f'(x)$, $\nu > 0$. The adjoint generalized heat equation is given by

$$\Delta_x u(x, t) + (\partial/\partial t)u(x, t) = 0.$$

The fundamental solution of the generalized heat equation is the function

$$G(x; t) = \int_0^{\infty} e^{-iu^2/2} \mathcal{g}(xu) d\mu(u) = \left(\frac{1}{2\pi}\right)^{1/2} e^{-x^2/4},$$

where

$$\mathcal{g}(z) = 2^{-1/2} \Gamma(\nu + \frac{1}{2}) z^{1/2 - \nu} J_{\nu - 1/2}(z),$$

$$d\mu(z) = \frac{1}{2^{\nu-1/2} \Gamma(\nu + \frac{1}{2})} z^{2\nu} d\zeta,$$

$J_{\nu}(z)$ being the ordinary Bessel function of order $\alpha$. If

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\[ D(x, y, z) = \frac{2^{\nu - \frac{1}{2}}[\Gamma(\nu + \frac{1}{2})]^2}{\Gamma(\nu)^{\frac{1}{2}}} \cdot (xyz)^{\nu - 2}[\Delta(x, y, z)]^{2 - \nu}, \]

where \( \Delta(x, y, z) \) is the area of a triangle whose sides are \( x, y, z \) if there is such a triangle, and otherwise, \( D(x, y, z) = 0 \), the associated function \( f(x, y) \) corresponding to a function \( f(x) \) is given by

\[ f(x, y) = \int_0^\infty f(u)D(x, y, u)du, \quad 0 < x, y < \infty. \]

The function associated with the fundamental solution \( G(x; t) \) is

\[ G(x, y; t) = \int_0^\infty e^{-t\xi}g(xu)g(yu)du = \left( \frac{1}{2t} \right)^{\nu+\frac{1}{2}}e^{-\left(\frac{x^2+y^2}{2t}\right)}I_{\nu+\frac{1}{2}}(\frac{xy}{2t}), \]

where

\[ I_{\nu}(x) = 2^{\nu-\frac{1}{2}}\Gamma(\nu + \frac{1}{2})\nu^{\frac{1}{2}}J_{\nu}(x), \]

\( J_{\nu}(x) \) being the Bessel function of imaginary argument and order \( \nu \).

A bounded function \( f(x) \) for which

\[ \sum_{j=1}^n \sum_{k=1}^n a_ja_kf(x_j, x_k) \geq 0 \]

for any \( x_1, x_2, \ldots, x_n > 0 \) and arbitrary complex numbers \( a_1, a_2, \ldots, a_n \) is said to be positive definite.

We establish two theorems, each a consequence of two basic lemmas which give integral representations for functions of one space variable.

**Lemma 1.** A function \( u(x, t) \) has the representation

\[ u(x, t) = \int_0^\infty e^{-tu^2}g(xu)du, \quad t > 0, \]

where \( \alpha(u) \) is a nondecreasing function iff

(i) \( \Delta_\alpha u(x, t) = -(\partial/\partial t)u(x, t), t > 0, \)

(ii) \( u(x, t) \geq 0, t > 0. \)

**Proof.** The necessity of the conditions is immediate on noting that \( g(xu) \geq 0 \) and that \( \Delta_\alpha g(xu) = u^2g(xu) \), with differentiation under the integral sign justifiable.

To establish sufficiency, consider the function

\[ v(x, t) = G(x; t)u\left(\frac{x}{t}, \frac{1}{t}\right), \quad t > 0. \]
Then it readily follows that $v(x, t)$ is a nonnegative generalized temperature function for $t > 0$. Hence by [2, p. 49], we have

$$v(x, t) = \int_0^\infty G(x, y; t) \, d\beta(y),$$

with $\beta(y)$ nondecreasing, or

$$u(x, t) = \int_0^\infty e^{-t \lambda^2} g(xy) \, d\alpha(y)$$

where $\alpha(y) = \beta(2y)$, and the proof is complete.

**Lemma 2.** A function $u(x, t)$ has the representation

$$u(x, t) = \int_0^\infty e^{-t \lambda^2} g(xy) \, d\alpha(y), \quad t > 0,$$

where $\alpha(u)$ is nondecreasing iff

(i) $\Delta u(x, t) = (\partial^2 / \partial t) u(x, t), \ t > 0,$
(ii) $u(x, t)$ is analytic for each $t > 0$ and $|\Re z| < R$,
(iii) $u(ix, t) \geq 0, \ t > 0$.

**Proof.** The necessity of the conditions is immediate with the analyticity of $u(z, t)$ a consequence of Theorem 5.3 of [2].

Conversely, since $u(iy, t) \geq 0$ and $u(iy, t)$ is a solution of the adjoint generalized heat equation for $t > 0$, the result follows by the preceding lemma.

Combining these two results, we have the following.

**Theorem 3.** A function $u(x, y, t)$ has the representation

$$u(x, y, t) = \int_0^\infty e^{-t \lambda^2} g(xy) \, d\mu(u), \quad t > 0,$$

with $\alpha(u)$ nondecreasing iff

(i) $\Delta u(x, y, t) = -\Delta v(x, y, t) = (\partial^2 / \partial t) u(x, y, t),$
(ii) for $y \geq 0, t > 0, u(z, y, t)$ is analytic for $|\Re z| < R$,
(iii) for each $x \geq 0, y \geq 0, t > 0, u(ix, y, t) \geq 0$.

**Proof.** The necessity of the conditions are readily verified. Conversely, an appeal to Lemma 1, yields, for fixed $x$,

$$u(x, y, t) = \int_0^\infty e^{-t \lambda^2} g(yu) \, d\varphi(x, u),$$

where, for each $x$, $\varphi(x, u)$ is nondecreasing. Hence
But, \( u(x, 0, t) \) satisfies the conditions of Lemma 2 so that

\[
\int_0^\infty e^{-tu^3}g(xu)da(u),
\]

with \( a(u) \) nondecreasing. Consequently,

\[
\int_0^\infty e^{-tu^3}g(xu)da(u) = d\varphi(x, u),
\]

and since the left-hand side is a Laplace transform, it follows by uniqueness, that, for each fixed \( x \),

\[
g(xu)da(u) = d\varphi(x, u),
\]

yielding the desired representation for \( u(x, y, t) \).

An example illustrating the theorem is given by

\[
\left( \frac{1}{2t} \right)^{+1/2} e^{-x^3+u^3/4u^3} \frac{xy}{2t} = \int_0^\infty e^{-tu^3}g(xu)g(yu)du(u).
\]

We establish criteria for a similar representation, but with \( a(u) \) a nondecreasing bounded function, again, by first proving two basic lemmas.

**Lem 4.** A necessary and sufficient condition that

\[
u(x, t) = \int_0^\infty e^{-tu^3}g(xu)da(u), \quad t > 0,
\]

with \( a(u) \) nondecreasing and bounded, is that

(i) \( \Delta^2 u(x, t) = (\partial/\partial t) u(x, t), \quad t > 0, \)

(ii) \( u(x, t) > -M \) for some \( M > 0, \quad t > 0, \)

(iii) \( u(x, 0+) \) exists and is positive definite.

**Proof.** If the integral representation holds, then (i) is immediate; (ii) follows from the fact that

\[
| u(x, t) | \leq \int_0^\infty da(u) < \infty, \quad t > 0;
\]

and (iii) from [1].

Conversely, we note that by (i) and (ii), \( u(x, t) + M \) is a positive generalized temperature function, and hence, by Corollary 8.6 of [2].
But by [1], (iii) implies that
\[ u(x, 0+) = \int_0^\infty \mathcal{J}(xu)\,d\alpha(u), \]
with \( \alpha(u) \) nondecreasing and bounded. Hence
\[
\begin{align*}
G(x, y; t) & d\beta(y) + \int_0^\infty \mathcal{J}(y)\,d\alpha(u) \\
& = \int_0^\infty \mathcal{J}(y)G(x, y; t)\,d\mu(y) \\
& = \int_0^\infty e^{-tu}\mathcal{J}(ux)\,d\alpha(u),
\end{align*}
\]
the interchange in order of integration being justifiable by Fubini's theorem, since \( \alpha(u) \) is a nondecreasing bounded function. Thus the proof is complete.

**Lemma 5.** A necessary and sufficient condition that
\[
u(x, t) = \int_0^\infty e^{-iu}\mathcal{J}(xu)\,d\alpha(u), \quad t > 0,
\]
with \( \alpha(u) \) nondecreasing and bounded is that
(i) \( \Delta u(x, t) = -(\partial/\partial t)u(x, t), \quad t > 0, \)
(ii) \( u(x, t) \) is analytic for each \( t > 0 \) and \( \Re z < R, \)
(iii) \( u(ix, t) > -M \) for some \( M > 0, \quad t > 0, \)
(iv) \( u(ix, 0+) \) exists and is positive definite.

**Proof.** The necessity of the conditions are immediate and the sufficiency follows on noting that \( u(x, -t) \) is a generalized temperature for \( t < 0, \) and hence \( u(ix, t) \) is a generalized temperature for \( t > 0. \) Thus \( u(ix, t) \) satisfies the conditions of Lemma 4 and hence the desired integral representation follows for \( u(x, t). \)

Lemmas 4 and 5 yield the following result whose proof is established analogously to that for Theorem 3 and hence will be omitted.

**Theorem 6.** A function \( u(x, y, t) \) has the representation
\[
u(x, y, t) = \int_0^\infty e^{-iu}\mathcal{J}(xu)\,d\alpha(u), \quad t > 0,
\]
with \( \alpha(u) \) nondecreasing and bounded iff
\[ \Delta_u(x, y, t) = -\Delta_u(x, y, t) = \frac{\partial^2}{\partial t^2}u(x, y, t), \quad t > 0. \]

(ii) For \( x \geq 0, \ t > 0 \), \( u(x, z, t) \) is analytic for \( |\text{Re } z| < R \),

(iii) \( u(x, iy, t) > -M \),

(iv) \( u(x, 0, 0+) \), \( u(0, iy, 0+) \) exist and are positive definite.

The theorem is illustrated by the function

\[ u(x, y, t) = e^{-a^2}g(xa)g(ya) \]

which satisfies the conditions of the theorem and which has the representation

\[ u(x, y, t) = \int_0^\infty e^{-ta^2}g(xu)g(yu)\,da(u), \]

where \( a(u) \) is constant except for a unit positive jump at \( u = a \).

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