EFFECTIVELY MINIMIZING EFFECTIVE FIXED-POINTS

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Abstract. This note answers an open problem posed by H. Rogers, Jr. on p. 202 of Theory of recursive functions and effective computability by proving the following invariant form of one of his results [op. cit., p. 200, Theorem XIV]: for any fixed-point function \( \zeta \) there exists a recursive function \( g \) such that if \( z \) is an index of an effective operator \( \Psi \), then \( g(z) \) is also an index of \( \Psi \), and \( \zeta(g(z)) \) is an index of the minimum fixed-point of \( \Psi \) with respect to inclusion.

Let \( W \) and \( \varphi \) denote standard enumerations of the recursively enumerable sets and the partial recursive functions, respectively. Let \( \zeta \) be a recursive fixed-point function for \( W \) (i.e. if \( \zeta(z) \) is in the domain of \( \varphi \), then \( \zeta(z) \) and \( \varphi \zeta(z) \) are \( W \)-indices of the same set, while otherwise \( \zeta(z) \) is a \( W \)-index of the empty set). Clearly, if \( z \) is an index of an effective operator \( \Psi \) (i.e. for all \( x \), \( \varphi_x(x) \) is a \( W \)-index of \( \Psi(W_x) \)), then \( \zeta(z) \) is a \( W \)-index of a fixed-point of \( \Psi \); unfortunately, it is not always a \( W \)-index of the minimum fixed point of \( \Psi \) (cf. [1, p. 196]).

However, there exists a recursive function \( g \) such that if \( z \) is an index of an effective operator \( \Psi \), then

(i) \( g(z) \) is also an index of \( \Psi \), and

(ii) \( \zeta g(z) \) is a \( W \)-index of the minimum fixed-point of \( \Psi \).

Proof. Let \( f \) be a recursive function whose range is \( \{ \langle x, y \rangle \mid y \in W_x \} \).

Abusing language, we say that \( y \) is in \( W_x \) before \( y' \) is in \( W_{x'} \) if there exists a number \( n \) such that \( \langle x, y \rangle \in \{ f(1), \ldots, f(n) \} \) but \( \langle x', y' \rangle \notin \{ f(1), \ldots, f(n) \} \).

We use the Recursion Theorem to obtain recursive functions \( k, h, \) and \( g \) such that

\[
W_k(z, x) = \{ y \mid \text{y is in } W_{h(z)} \text{ before } x \text{ is in } W_{\varphi_k(z, x)} \},
\]

\[
W_h(z) = \{ x \mid x \text{ is in } W_{\varphi_k(z, x)} \text{ before } x \text{ is in } W_h(z) \},
\]

\[
\varphi_h(z) = \lambda x [\varphi_h(x) \text{ if } x \neq \zeta g(z); h(z) \text{ if } x = \zeta g(z)].
\]

Now suppose that \( z \) is an index of an operator \( \Psi \). Then \( \Psi(W_h(z)) \subset W_k(z) \), since if \( x \in \varphi(W_h(z)) \), then \( x \in \Psi(W_k(z)) = W_{\varphi_k(z, x)} \) (for otherwise \( W_k(z, x) = W_h(z) \) while \( x \in \Psi(W_h(z)) \)) and hence \( x \in W_h(z) \).
(for otherwise we violate the definition of $W_{h(z)}$). Conversely, 
$W_{h(z)} \subseteq \Psi(W_{h(z)})$ for if $x \in W_{h(z)}$, then $x \in W_{\varphi_{h(z,x)}} = \Psi(W_{h(z,x)})$ 
and hence $x \in \Psi(W_{h(z)})$ (since $W_{h(z,x)} \subseteq W_{h(z)}$ and $\Psi$ is monotone by 
the Myhill-Shepherdson Theorem [1, p. 395]). Thus $W_{h(z)}$ is a fixed-
point of $\Psi$.

To see that $W_{h(z)}$ is the minimum fixed-point of $\Psi$, suppose that 
$A$ is a fixed-point of $\Psi$, that $x \in W_{h(z)}$, and that all numbers which 
are in $W_{h(z)}$ before $x$ are members of $A$. Then $W_{h(z,x)} \subseteq A$, for if 
y \in $W_{h(z,x)}$, then $y$ is in $W_{h(z)}$ before $x$ is in $W_{\varphi_{h(z,x)}}$ which is before $x$ 
is in $W_{h(z)}$. Thus $\Psi(W_{h(z,x)}) \subseteq \Psi(A) = A$. But, as already noted, 
x \in $\Psi(W_{h(z,x)})$ whenever $x \in W_{h(z)}$. Thus $x \in A$. So, by induction, 
$W_{h(z)} \subseteq A$.

Thus (ii) is satisfied since by our choice of $g,$

\[ W_{\tilde{g}(z)} = W_{\varphi_{g(z)}}(\tilde{g}(z)) = W_{h(z)}. \]

We see that (i) is also satisfied, since in the case where $x = \tilde{g}(z),$

\[ W_{\varphi_{g(z)}}(z) = W_{\varphi_{g(z)}}(\tilde{g}(z)) = W_{\tilde{g}(z)} = \Psi(W_{\tilde{g}(z)}) \]
\[ = W_{\varphi_{g(z)}}(x) = W_{\varphi_{g(z)}} = \Psi(W_{z}). \]

while in the case where $x \neq \tilde{g}(z), \varphi_{g(z)}(x) = \varphi(x)$ and hence 
$W_{\varphi_{g(z)}}(x) = W_{\varphi_{g(z)}} = \Psi(W_{x}).$ \[ \square \]

It should be noted that if one identifies partial recursive functions 
with their graphs, then the above discussion remains valid if “$W$” 
is replaced by “$\varphi$”.

References

1. Hartley Rogers, Jr., Theory of recursive functions and effective computability, 

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