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p-SOLVABLE GROUPS WITH FEW AUTOMORPHISM CLASSES OF SUBGROUPS OF ORDER p

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ABSTRACT. This paper investigates the relationship between the p-length, $l_p(G)$, of the finite p-solvable group G and the number, $a_p(G)$, of orbits in which the subgroups of order p are permuted by the automorphism group of G. If $p>2$ and $a_p(G)\leqq 2$, it is shown that $l_p(G)\leqq a_p(G)$. If $p=2$ and $a_2(G)=1$, it is proved that either $l_p(G)\leqq a_p(G)$ or $G/O_{2'}(G)$ is a specific group of order 48.

1. **Introduction.** If p is a prime, G is a finite p-solvable group, and the automorphisms of G permute the subgroups of G of order p in exactly $a_p(G)$ orbits (this defines $a_p(G)$), what, if any, is the relationship between $a_p(G)$ and $l_p(G)$, the p-length of G? If p is odd and $a_p(G)=1$, then Shult [6] proved that $l_p(G)=1$. Thompson [7, Lemma 5.40] showed that $l_2(G)=1$ if a Sylow 2-group of G contains more than one involution and the *inner* automorphisms of G transitively permute the involutions of G.

The main results of the present paper are:

(1) If G is a finite p-solvable group, $p>2$, and $a_p(G)=2$, then $l_p(G)\leqq 2$;

(2) If G is a finite solvable group and $a_2(G)=1$, then either $l_2(G)=1$ or $G/O_{2'}(G)$ is a specific group of order 48 and $l_2(G)=2$.

The upper bound in (1) is best possible as is shown by the following example. Let p be a prime, let V be the additive group of the field $\mathrm{GF}(p^p)$, and let λ be a generator of the multiplicative group of $\mathrm{GF}(p^p)$. Define the automorphisms A and B of V by $xA=\lambda x$ and $xB=x^p$ for $x\in V$. A has order p^p-1, B has order p, and $B^{-1}AB=A^p$. If G is the semidirect product of V and the group generated by A and B, then G is solvable, $a_p(G)=2$ (in fact the inner automorphisms of G permute the subgroups of order p in 2 orbits), and $l_p(G)=2$.

2. **Notation and preliminary results.** The concepts of p-solvability and p-length are defined in [4]. It should be noted that a 2-solvable

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group is solvable because of the Feit-Thompson Theorem [2]. All group characters considered in this paper are over the field of complex numbers. If χ is a character of G, then $e_\chi = (\sum_{x\in G} \chi(x^2))/|G|$. It is known [1, Theorem 3.5] that $e_\chi = 1$, -1, or 0 depending on whether χ is the character of a real representation, χ is real valued, but is not the character of a real representation, or χ is not real valued, respectively. The rest of the notation used agrees with [2].

LEMMA 2.1 (THOMPSON [7, LEMMA 5.17]). *Let G be a finite group, P a normal p-subgroup of G, and Q a subgroup of G of order not divisible by p. If $[P, Q] = P \neq 1$, and Q centralizes every characteristic abelian subgroup of P, then P is a nonabelian special p-group.*

LEMMA 2.2. *Let G be a finite group, P a normal p-subgroup of G, Q a subgroup of G of order not divisible by p, and $C = C_P(Q)$. Assume either $p > 2$ or $P' = 1$. If every element of order p in P is conjugate in P to an element of C, then $C = P$.*

PROOF. If $C \neq P$, then there is an integer n such that $Z_n(P) \leqq C$, but $Z_{n+1}(P) \nleqq C$. Let x be any element of order p in $Z_{n+1}(P)$. Then $x^y \in C$ for some $y \in P$. Since $x^y \equiv x \pmod{Z_n(P)}$, it follows that $x \in C$. A theorem of Huppert [5] now implies that $Z_{n+1}(P) \leqq C$.

LEMMA 2.3. *Let G be the group with generators x and y and relations $x^4 = y^3 = x^2(xy)^4 = 1$. Then $|G| = 48$, $|Z(G)| = 2$, $G/Z(G)$ is isomorphic to S_4, $l_2(G) = 2$, and a Sylow 2-group of G is generalized quaternion.*

PROOF. $G = \langle x, xy\rangle$ and $x^2 = (xy)^4$. Thus $x^2 \in Z(G)$. $G/\langle x^2\rangle$ is generated by two elements $\bar{x}$ and $\bar{y}$ where $\bar{x}^2 = \bar{y}^3 = (\bar{x}\bar{y})^4 = 1$. It follows that $G/\langle x^2\rangle$ is a homomorphic image of S_4. Therefore, $|G| \leqq 48$. Now it is verified easily that the permutations

$$(1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16)$$

and

$$(2, 5, 9)(4, 7, 11)(6, 12, 13)(8, 10, 15)$$

satisfy our relations and generate a group of order at least 48. Hence $|G| = 48$ and $G/\langle x^2\rangle$ is isomorphic to S_4. Since $Z(S_4) = 1$, this implies that $Z(G) = \langle x^2\rangle$. It is possible to choose the homomorphism of G onto S_4 such that x is mapped onto (12) and y is mapped onto (134). Then if z is any involution in G, z must be conjugate in G to an element of either $Z(G)x$ or $Z(G)(xy)^2$. It now follows that x^2 is the only involution in G. Hence a Sylow 2-group of G is generalized quaternion.

LEMMA 2.4. *Let G be a finite solvable group of 2-length >1 such that a Sylow 2-group of G has only one involution. Assume that $O_{2'}(G)=1$. Then $l_2(G)=2$, a Sylow 2-group of G is generalized quaternion, $|Z(G)|=2$, $G/Z(G)$ is isomorphic to S_4, and G is isomorphic to the group with generators x and y and relations $x^4=y^3=x^2(xy)^4=1$.*

PROOF. Let $P=O_2(G)$. Then $C_G(P)=Z(P)$ [4, Lemma 1.2.3], and G/P is not a 2-group but has order divisible by 2. Thus $G/Z(P)$ is isomorphic to a subgroup of the automorphism group of P. Since P has only one involution, P is either cyclic or generalized quaternion. It now follows that P must be quaternion of order 8 (otherwise $G/Z(P)$ would have to be a 2-group). The automorphism group of P then is isomorphic to S_4 and $Z(P)=Z(G)$. Since $l_2(G/Z(P))=l_2(G)>1$, it follows that $G/Z(G)$ is isomorphic to S_4. Then $|G|=48$. It only remains to show that G is generated by two elements satisfying the above relations.

Now S_4 is generated by the two permutations (12) and (134). Thus G contains two elements x and y such that y has order 3, $G/Z(G)$ is generated by $xZ(G)$ and $yZ(G)$, and $xZ(G)$ and $(xy)Z(G)$ have orders 2 and 4, respectively, in $G/Z(G)$. Since G has only one element of order 2, it follows that $G=\langle x, y\rangle$ and $x^4=y^3=x^2(xy)^4=1$.

THEOREM 2.5. *Let P be a 2-group of exponent 4 such that $a_2(P)=1$. Assume $1<r=|\Omega_1(P)|<|P|$. Then $\Omega_1(P)=D(P)\leqq Z(P)$, $Z(P)$ is either P or $\Omega_1(P)$, and P' is either 1 or $\Omega_1(P)$. If χ is an irreducible character of P whose kernel does not contain $\Omega_1(P)$, then one of the following must be satisfied:*

(a) *$|P|=r^2$ and $e_\chi=0$.*

(b) *P is a nonabelian special 2-group of order r^3, $\chi(1)=r$, $e_\chi=-1$, and $|C_P(x)|=r^2$ if x is any element of order 4 in P.*

PROOF. Since $e(P)=4$, $P/\Omega_1(P)$ has exponent 2 and so is abelian so $P'\leqq\Omega_1(P)$. Since $D(P)$ is generated by commutators and squares (which lie in $\Omega_1(P)$), $D(P)\leqq\Omega_1(P)$. Since the automorphisms of P transitively permute the involutions in P, we must have $D(P)=\Omega_1(P)\leqq Z(P)$. Now let $M=D(P)$. If $x\in M^\#$, let m be the number of solutions in P to $y^2=x$. m is independent of x and so $|P|=m(r-1)+r$. Hence $(r-1)$ divides $(|P|-1)$. Since r and $|P|$ are both powers of 2, this implies that $|P|=r^n$ for some integer $n>1$ and then $m=(r^n-r)/(r-1)$. A similar argument yields that $|Z(P)|$ is a power of r. Now if either $|Z(P)|>r^2$ or $|Z(P)|=r^2<r^n$, then there are elements x and y in $P-M$ such that $Mx\neq My$, $[x, y]=1$, and $x^2=y^2$.

But then $(xy)^2=1$ is contrary to $xy\notin M$. Hence either $Z(P)=M$ or $|Z(P)|=|P|=r^2$. Thus if P is abelian, $n=2$, and we see that P is the direct product of several cyclic groups of order 4. Then, since M is not in the kernel of χ, χ cannot be real valued and so $e_\chi=0$.

Now suppose $P'\neq 1$. Then $P'\cap M\neq 1$ and so $P'\geqq M=D(P)\geqq P'$. Hence $P'=M$. Let N be the intersection of M and the kernel of χ. N must have order $r/2$. $\chi(x^2)=\chi(1)$ if $x^2\in N$, and since $M/N \leqq Z(P/N)$, $\chi(x^2)=-\chi(1)$ if $x^2\notin N$. It now follows that

$$\begin{aligned} r^n e_\chi &= \chi(1)[m(|N| - 1) + r - m(|M| - |N|)] \\ &= -\chi(1)(r^n - r^2)/(r-1). \end{aligned}$$

If $n=2$, then $e_\chi=0$. Suppose now $n>2$. Then we must have $e_\chi=-1$ and $\chi(1)=r^{n-2}(r-1)/(r^{n-2}-1)$. For this to be an integer, we must have $n=3$. Then $\chi(1)=r$. Since $|P|\neq r^2$, $Z(P)=M$. P now is a special nonabelian 2-group.

Finally let x be an element of order 4 in P. Then all conjugates of x are contained in Mx. Hence $|P:C_P(x)|\leqq r$. If $|P:C_P(x)|<r$, then P must have more than $r+(r^3-r)/r=r^2+r-1$ conjugacy classes. Since P has exactly r^2 linear characters, this would imply that P has at least r irreducible characters of degree r. Since $r^2+r(r)^2>|P|$, this is impossible. Hence $|P:C_P(x)|=r$ and so $|C_P(x)|=r^2$.

3. The main results.

THEOREM 3.1. *If G is a finite p-solvable group for the odd prime p and $a_p(G)=2$, then $l_p(G)\leqq 2$.*

PROOF. Let G be a counterexample to the theorem of minimal order. Then $O_{p'}(G)=1$. Let A be the holomorph of G, let $S=O_p(G)$, and let H be a Hall p'-subgroup of $O_{pp'}(G)$. The Frattini argument implies that $A=SN_A(H)$. Let $M=S\cap N_A(H)$. If $M=1$, it follows easily that any two subgroups of order p in $N_G(H)$ are conjugate in $N_A(H)$. Thus $a_p(G/S)\leqq 1$. This implies that $l_p(G/S)\leqq 1$ [6]. Since $l_p(G/S) =l_p(G)-1$, this is a contradiction.

Hence $M\neq 1$. $M=C_S(H)$ and so, from Lemma 2.2, not every element of order p in S can be conjugate in S to an element of M. Since $A=SN_A(H)=SN_A(M)$, it follows that there is an element of order p in S which is not conjugate in A to any element of M. Since $a_p(G)=2$, this implies that S contains all elements of order p in G and that any two subgroups of order p in M are conjugate in A. If it were true that all subgroups of order p in M were conjugate in $N_A(H)$, then it would follow that $a_p(N_G(H))=1$. But this would imply that $l_p(G)=l_p(G/S)+1\leqq 2$.

Thus there are two subgroups P_1 and P_2 of order p in M that are not conjugate in $N_A(H)$. We must have $P_2 = P_1^{xy}$ for some $x \in N_A(H)$ and some $y \in S$. Let z be a generator of P_1. Since $P_2 \neq P_1^x$, we obtain $1 \neq [z^x, y] \in M \cap [\Omega_1(M), S]$. But if P is a subgroup of order p contained in $M \cap [\Omega_1(M), S]$, then every subgroup of order p in M is conjugate in A to P. Since $[\Omega_1(M), S] \lhd SN_A(M) = A$, it follows that $[\Omega_1(M), S] \geqq \Omega_1(M)$. Since S is nilpotent, we have a contradiction and the theorem is proved.

THEOREM 3.2. *Let G be a finite solvable group such that $a_2(G) = 1$. Then either $l_2(G) \leqq 1$ or $l_2(G) = 2$ and $G/O_{2'}(G)$ is a group of order* 48 *which is isomorphic to the group with generators x and y and relations $x^4 = y^3 = x^2(xy)^4 = 1$.*

PROOF. Suppose G is a counterexample to the theorem of minimal order. Because of Lemma 2.4, a Sylow 2-group of G must have more than one involution. It now follows that $O_{2'}(G) = 1$ and $G = O_{22'2}(G)$. Now let A be the holomorph of G. Then $G \lhd A$ and all involutions of G are conjugate in A. Let $S = O_2(G)$, $M = \Omega_1(Z(S))$, $Q =$ a Hall $2'$-subgroup of G, $P =$ a Sylow 2-subgroup of $N_G(Q)$, and $N = N_A(Q) \cap N_A(P)$. Then the Fitting argument yields $G = SQP$, $A = NG$, $N_G(Q) = PQ$, and $N_A(Q) = NPQ$.

Since $l_2(G) > 1$, we must have $G \neq SQ$. Since $M \lhd A$, M must contain all involutions in G. Now $1 \neq M \cap N_A(Q) \lhd A$ since $A = N_A(Q)S$ and $M \leqq Z(S)$. It follows that $M \leqq N_A(Q)$. This implies that $M \leqq P$ and $[M, Q] = 1$. Now $M \cap Z(P) \neq 1$ and $M \cap Z(P) \lhd SQN_A(P) = A$. Thus $M \leqq Z(P)$. It follows that $M \leqq Z(G)$ and so conjugation by elements of N transitively permutes the nonidentity elements of M.

Suppose K is a proper subgroup of Q which is normalized by PN. Then SKP satisfies the hypothesis of the theorem and has smaller order than G. Thus, $l_2(SKP) = 1$ which implies that $[P, K] = 1$. It now follows (see, for example, [**3**, Lemma 2.4]) that Q is a special q-group for some odd prime q, NP transforms Q/Q' irreducibly, and $C_Q(P) = Q'$.

Since Q centralizes M, Q must centralize any abelian 2-subgroup of G which is normalized by Q. Suppose now $L = [S, Q]$. $1 \neq L \leqq Q$ and $L \lhd A$. Thus L must contain M. Now $L = [L, Q]$ and Q centralizes any characteristic abelian subgroup of L. Thus, by Lemma 2.1, L is a nonabelian special 2-group. It follows that $L' = M$. Since $L/M = (C_L(Q)/M) \times ([L/M, Q])$, we have $C_L(Q) = M$. Suppose now $L < S$. Then S contains a subgroup R such that $L < R$ and R/L is a minimal normal subgroup of A/L. Since $[R, Q] = L$, $R = LC_R(Q)$ and so $C_R(Q)$ is not contained in L. Now let $T/M = C_{L/M}(R)$. Then $T \lhd A$

and $T>M$. Since $[C_R(Q), T, Q]\leqq[M, Q]=1$ and $[Q, C_R(Q), T]=1$, the three subgroups lemma implies that $[T, Q, C_R(Q)]=1$. But $T=C_T(Q)[T, Q]=M[T, Q]$. It now follows that $[T, C_R(Q)]=1$. Let $x\in C_R(Q)-L$. Then $x^2\neq 1$, but $C_R(Q)/(L\cap C_R(Q))$ is isomorphic to the elementary abelian group R/L. Hence $x^2\in L\cap C_R(Q)=C_L(Q)=M$. Now $\{y^2|y\in T-M\}$ is a subset of $M^\#$ normalized by A. Hence there is a y in $T-M$ such that $y^2=x^2$. Since $[C_R(Q), T]=1$, $(xy)^2=x^2y^2=1$ contrary to $xy\notin M$. Thus $L=S$. Therefore S is a nonabelian special 2-group, $S'=M$, $[S, Q]=S$, and $C_S(Q)=M$. Since $S\cap P\leqq C_S(Q)$, we have $S\cap P=M$.

Next let $r=|M|$. $r>2$ and, by Theorem 2.5, $|S|=r^2$ or r^3. If $|S|=r^2$ and $y\in M^\#$, then the solutions in S to $x^2=y$ will constitute a single coset of M. Since $[M, Q]=1$, this would imply that $[S/M, Q]=1$. Since this contradicts $[S, Q]=S$, we must have $|S|=r^3$. Then, by Theorem 2.5, $|C_S(x)|=r^2$ for all $x\in S-M$.

If $M<H<P$ and N normalizes H, then SQH is a counterexample to the theorem of smaller order than G. Hence G/SQ is a minimal normal subgroup of A/SQ. Therefore P/M is elementary abelian. The elements of N induce automorphisms of P which transitively permute the involutions in P. Hence $|P|=r^n$ where $n=2$ or 3.

Now let $C/M=C_{S/M}(P)$. $M<C<S$ since $C_{G/M}(S/M)=S/M$ [4, Lemma 1.2.5]. CP satisfies the hypothesis of Theorem 2.5 and so $|CP|=|C/M||P|=r^2$ or r^3. It now follows that $|P|=|C|=r^2$, CP is a nonabelian special 2-group of order r^3, and $|C_{CP}(x)|=r^2$ if $x\in CP-M$. Now $M<[S, P]<S$ and $[S, P]$ is normalized by N. Theorem 2.5 now implies that $|[S, P]|=r^2$. Since N normalizes $C\cap[S, P]$ and $C\cap[S, P]>M$, we obtain, using Theorem 2.5 again, that $|C\cap[S, P]|=r^2$. Thus $C=[S, P]$.

Since $[P, C, S]\leqq[M, S]=1$ and $[C, S, P]\leqq[M, P]=1$, the three subgroups lemma implies $C'=[C, C]=[S, P, C]=1$. An immediate consequence of this is that $C_S(x)=C_{CP}(x)=C$ for all $x\in C-M$. It follows from this that $C_C(x)=M$ if $x\in S-C$ or if $x\in P-M$. Q centralizes any abelian 2-subgroup of G which it normalizes and $C_S(Q)=M$. Thus Q does not normalize C. Since $N_A(C)\geqq SPN$, we must have $|A:N_A(C)|=|Q:N_Q(C)|>1$. Now distinct conjugates of C have only the elements of M in common. This implies that $|Q:N_Q(C)|\leqq r+1$. Now $N_Q(C)$ is normalized by PN, $N_Q(C)\neq Q$, and $N_Q(C)\geqq C_Q(P)=Q'$. This implies that $N_Q(C)=Q'$. Therefore $|Q/Q'|\leqq r+1$.

If $r=4$, then $|Q/Q'|=3$ or 5 and so $Q'=1$. But P/M, which is elementary abelian of order r, is faithfully represented as a group of automorphisms of Q. Since this is impossible, we now assume $r>4$.

Now let $H=SP$. H has exponent 8, $H'=D(H)=C$, and $Z(H)=M$. We now proceed to count the number of elements of order 4 in H. If $x\in P-M$ and $y\in S$, then $(xy)^2\equiv[x, y]$ (mod M). Thus $(xy)^2\in M$ if, and only if, $yM\in C_{S/M}(x)$. Let $K/M=C_{S/M}(x)$ and let $L=C_S(x)$. If $z\in M$, then $L=C_S(xz)$. Since P/M is abelian, P normalizes L. $L\cap C=C_C(x)=M$ and so $[L, P]\leqq L\cap C=M$. Thus $L/M\leqq C/M$ which can happen only if $L=M$. Now the mapping which maps y to $[y, x]$ is a homomorphism of K into M with kernel L. Hence $|K|\leqq r^2=|C|$. Therefore $K=C$. Thus we have shown that if $x\in P-M$ and $y\in S$, then $(xy)^2\in M$ if, and only if, $y\in C$.

The elements of H of order dividing 4 are precisely those elements which belong to either CP or S. Hence H has exactly $|S|+|CP|-|C|=2r^3-r^2$ elements of order dividing 4. The number of elements of order 4 in H then must be $2r^3-r^2-r$. Since all involutions are conjugate, if $y\in M^\#$, there must be exactly $(2r^3-r^2-r)/(r-1)=2r^2+r$ solutions in H to $x^2=y$.

Now let χ be an irreducible character of H whose kernel does not contain M, and let K be the intersection of M and the kernel of χ. Then $|K|=r/2$, $\chi(z)=\chi(1)$ if $z\in K$, and $\chi(z)=-\chi(1)$ if $z\in M-K$.

If $x\in C-M$ and $z\in M-K$, then $\chi(xz)=-\chi(x)$. But all conjugates of x in S belong to Mx and $|S:C_S(x)|=r$. Thus x and xz are conjugate in S. This implies that χ has the value 0 on $C-M$. We now obtain

$$\begin{aligned} r^4e_\chi &= \sum_{x\in H;\,x^2\in M}\chi(x^2)\\ &= \chi(1)[r+(2r^2+r)(|K| -1-(|M| - |K|))]\\ &= -\ 2r^2\chi(1). \end{aligned}$$

Thus $\chi(1)=r^2/2$.

The sum of the squares of the degrees of the irreducible characters of H whose kernels do contain M must be $|H/M|=r^3$. Therefore the number of irreducible characters of H whose kernels do not contain M is $(r^4-r^3)/(r^2/2)^2=4(r-1)/r$. Since $r>4$, this cannot be an integer and so the theorem is proved.

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