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THE FUNCTIONAL EQUATION OF SOME DIRICHLET SERIES. II

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ABSTRACT. We derive the functional equation of a class of Dirichlet series. A particular case of our result was first given by Rademacher.

For any positive integer k, Rademacher [2] showed that the Dirichlet series

$$Z(s) = \sum_{l=1}^{k} \left\{ \sum_{n>0;\, n\equiv l(k)} n^{-s} + \sum_{n>0;\, n\equiv -l(k)} n^{-s} \right\}^2 \qquad (\sigma = \operatorname{Re} s > 1)$$

has an analytic continuation to the entire complex s-plane that is analytic except for a double pole at $s = 1$, and satisfies the functional equation

$$(1) \qquad (\pi/k)^{-s}\Gamma^2(s/2)Z(s) = (\pi/k)^{s-1}\Gamma^2(\{1-s\}/2)Z(1-s).$$

Rademacher's proof used a familiar representation of the Hurwitz zeta-function. The purpose of this note is to show that a simpler proof of (1) as well as a considerable generalization can be given by employing Epstein zeta-functions rather than the Hurwitz zeta-function.

For g and h real and $\sigma > 1$ let

$$Z(s; g, h) = \sum_{n}{}' e^{2\pi i h n} |n+g|^{-s},$$

where the dash ′ indicates that the summation is over all integers n except in the possibility that $n + g = 0$. $Z(s; g, h)$ has an analytic continuation to the entire complex plane and is entire if h is not an integer and is analytic everywhere except at $s = 1$ where there is a simple pole with residue 2 when h is an integer [1]. Furthermore, [1, p. 207] we have the functional equation

$$(2) \quad \pi^{-s/2}\Gamma(s/2)Z(s; g, h) = e^{-2\pi i g h}\pi^{(s-1)/2}\Gamma(\{1-s\}/2)Z(1-s; h, -g).$$

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Now, for any fixed positive integer k and h real, define for all s,

$$Z_h(s) = \sum_{l=1}^{k} e^{4\pi i l h/k} Z^2(s; l/k, h)$$

and

$$Z_h^*(s) = \sum_{l=1}^{k} Z(s; (l+h)/k, 0) Z(s; (k-l+h)/k, 0).$$

We shall now prove the

THEOREM. *$Z_h(s)$ and $Z_h^*(s)$ satisfy the functional equation*

$$(3) \qquad (\pi k)^{-s}\Gamma^2(s/2)Z_h(s) = (\pi k)^{s-1}\Gamma^2(\{1-s\}/2)Z_h^*(1-s).$$

For all h, $Z_h^*(s)$ has a double pole at $s = 1$. If k is even and $h \equiv 0 \pmod{\frac{1}{2}k}$, or if k is odd and $h \equiv 0 \pmod{k}$, then $Z_h(s)$ has a double pole at $s = 1$. If h is an integer not satisfying either of the former conditions, $Z_h(s)$ is either entire or has a simple pole at $s = 1$. If h is not an integer, $Z_h(s)$ is entire.

PROOF. Consider (2) with $g = l/k$, $1 \leqq l \leqq k$, and $\sigma < 0$. Put $n = mk + j$, $-\infty < m < \infty, j = 1, \cdots, k$. Then,

$$\begin{aligned}&\pi^{-s/2}\Gamma(s/2)e^{2\pi i l h/k}Z(s; l/k, h)\\ &\quad = \pi^{(s-1)/2}\Gamma(\{1-s\}/2)\sum_n{}' e^{-2\pi i l n/k}|n+h|^{s-1}\\ &\quad = \pi^{(s-1)/2}\Gamma(\{1-s\}/2)\sum_{j=1}^{k} e^{-2\pi i l j/k}\sum_m{}' |mk+j+h|^{s-1},\end{aligned}$$

or

$$(4) \qquad \begin{aligned}&(\pi k)^{-s/2}\Gamma(s/2)e^{2\pi i l h/k}Z(s; l/k, h)\\ &\quad = (\pi k)^{(s-1)/2}\Gamma(\{1-s\}/2)k^{-1/2}\sum_{j=1}^{k} e^{-2\pi i l j/k}Z(1-s; (j+h)/k, 0).\end{aligned}$$

Define

$$\xi(s; l/k, h) = (\pi k)^{-s/2}\Gamma(s/2)e^{2\pi i l h/k}Z(s; l/k, h),$$

the symmetric $k \times k$ matrix $A = [a_{lj}] = [k^{-1/2}e^{-2\pi i l j/k}]$, and $v_r(s, h)$ to be the column vector whose lth component is $\xi(s; (l+r)/k, h)$, $1 \leqq l \leqq k$. Then the k relations given by (4) can be written as

$$(5) \qquad v_0(s, h) = Av_h(1-s, 0).$$

Now, $A^2 = H = k^{-1}[b_{lj}]$, where

$$\begin{aligned} b_{lj} = \sum_{m=1}^{k} e^{-2\pi i(l+j)m/k} &= k, \quad \text{if } l + j = k \text{ or } 2k,\\ &= 0, \quad \text{otherwise,}\end{aligned}$$

i.e., if $H = [h_{lj}]$, $h_{lj} = 0$ except when $l + j \equiv 0 \pmod{k}$ in which case $h_{lj} = 1$. If T denotes the transpose, we then have from (5)

$$\begin{aligned}(\pi k)^{-s}\Gamma^2(s/2)Z_h(s) &= v_0^T(s, h)v_0(s, h) = \{Av_h(1-s, 0)\}^T Av_h(1-s, 0)\\ &= v_h(1-s, 0)^T H v_h(1-s, 0) = (\pi k)^{s-1}\Gamma^2(\{1-s\}/2)Z_h^*(1-s),\end{aligned}$$

by a direct calculation and the fact that $Z(s; h/k, 0) = Z(s; (h+k)/k, 0)$. This then proves (3).

It is clear from our remarks on $Z(s; g, h)$ that $Z_h^*(s)$ has a double pole at $s = 1$. Also, if $h \equiv 0 \pmod{\frac{1}{2}k}$ when k is even, or if $h \equiv 0 \pmod{k}$ when k is odd, the coefficient of $(s-1)^{-2}$ in the Laurent expansion of $Z_h(s)$ about $s = 1$ is easily seen to be $4k$. However, for other integral values of h, the coefficient of $(s-1)^{-2}$ is

$$\sum_{l=1}^{k} 4e^{2\pi i l(2h)/k} = 0.$$

In general, the constant term in the Laurent expansion of $Z(s; g, h)$ about $s = 1$ is a function of g. Thus, $Z_h(s)$ may have a simple pole at $s = 1$ or might be analytic at $s = 1$. Since $Z(s; g, h)$ is entire if h is not an integer, then clearly $Z_h(s)$ is entire as well, and this completes the proof.

We now show that Rademacher's result (1) is a special case of (3). Put $h = 0$ in (3). It is readily seen that $Z(s; l/k, 0) = Z(s; (k-l)/k, 0)$. Hence $Z_0(s) = Z_0^*(s)$. Now for $\sigma > 1$,

$$\begin{aligned}Z_0(s) &= \sum_{l=1}^{k} Z^2(s; l/k, 0) = k^{2s}\sum_{l=1}^{k}\left\{\sum_m{}' |mk + l|^{-s}\right\}^2 = k^{2s}\sum_{l=1}^{k}\left\{\sum_{n\equiv l(k)}{}' |n|^{-s}\right\}^2\\ &= k^{2s}\sum_{l=1}^{k}\left\{\sum_{n>0;\, n\equiv l(k)} n^{-s} + \sum_{n>0;\, n\equiv -l(k)} n^{-s}\right\}^2 = k^{2s}Z(s),\end{aligned}$$

and hence (3) reduces to (1).

References

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