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MINIMAL SURFACES IN S^m WITH GAUSS CURVATURE $\leqq 0$

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ABSTRACT. Closed minimal surfaces in a unit m-sphere S^m with Gauss curvature $K \leqq 0$ are considered.

1. **Introduction.** Recently, S. S. Chern, M. do Carmo, and S. Kobayashi [2] studied the n-dimensional submanifolds of a unit m-sphere S^m with scalar curvature $\geqq n(n-1) - n(m-n)/(2m-2n-1)$. In particular, they proved that the only closed minimal surfaces of S^m with Gauss curvature $K \geqq (2m-6)/(2m-5)$ are the following surfaces:

(i) equatorial sphere of S^3,
(ii) Clifford torus in S^3, and
(iii) Veronese surface in S^4.

The main purpose of this paper is to study the closed minimal surfaces of S^m with Gauss curvature $K \leqq 0$.

2. **Preliminaries.**[2] Let M be a surface in a unit m-sphere S^m. We choose a local field of orthonormal frames $e_1, \cdots, e_m$ in S^m such that, restricted to M, the vectors e_1, e_2 are tangent to M (and, consequently, $e_3, \cdots, e_m$ are normal to M). With respect to the frame field of S^m chosen above, let $\omega_1, \cdots, \omega_m$ be the field of dual frames. Then the structure equations of S^m are given by

$$(1) \qquad d\omega_A = \sum \omega_B \wedge \omega_{BA}, \qquad \omega_{AB} + \omega_{BA} = 0,$$

$$(2) \qquad d\omega_{AB} = \sum \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B, \qquad A, B, C = 1, \cdots, m.$$

We restrict these forms to M. Then

$$(3) \qquad \omega_r = 0, \qquad r, s, t = 3, \cdots, m.$$

Since $0 = d\omega_r = \omega_1 \wedge \omega_{1r} + \omega_2 \wedge \omega_{2r}$, by Cartan's lemma we may write

$$(4) \qquad \omega_{ir} = h^r_{i1}\omega_1 + h^r_{i2}\omega_2, \quad h^r_{ij} = h^r_{ji}, \qquad i, j = 1, 2.$$

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[2] Manifolds, mappings, functions, and other geometric objects are assumed to be differentiable and of class C^∞.

From these we obtain

$$d\omega_i = \sum \omega_j \wedge \omega_{ji}, \tag{5}$$

$$d\omega_{12} = -\left\{1 + \sum_{r=3}^{m} \det (h_{ij}^r)\right\}\omega_1 \wedge \omega_2, \tag{6}$$

$$d\omega_{ir} = \sum \omega_{ij} \wedge \omega_{jr} + \sum \omega_{is} \wedge \omega_{sr}. \tag{7}$$

Put

$$H = \left(\frac{1}{2}\right)\sum_{r=3}^{m}(h_{11}^r + h_{22}^r)e_r. \tag{8}$$

Then H is a well-defined normal vector field over M, and is called the mean curvature vector of M in S^m. If $H = 0$ identically on M, then M is called a minimal surface of S^m. The Gauss curvature K of M is given by

$$K = 1 + \sum_{r=3}^{m} \det (h_{ij}^r). \tag{9}$$

Let $e = \sum_{r=3}^m \cos \theta_r e_r$ be a unit normal vector at p; then the Lipschitz-Killing curvature $G(p, e)$ with respect to e is given by

$$G(p, e) = \left(\sum_r \cos \theta_r h_{11}^r\right)\left(\sum_s \cos \theta_s h_{22}^s\right) - \left(\sum_t \cos \theta_t h_{12}^t\right)^2. \tag{10}$$

Let ∇' be the covariant differentiation on S^m, and η be a normal vector field over M in S^m. If the covariant differentiation $\nabla'\eta$ has no normal component, then η is said to be *parallel in the normal bundle*. A unit normal vector field $\bar{e}$ over M is called a *minimal direction* if the Lipschitz-Killing curvature with respect to $\bar{e}$ is minimal at every point $p \in M$, i.e. $G(p, \bar{e}) = \min \{G(p, e): e$ unit normal vector at $p\}$, for all $p \in M$.

THEOREM 1. *Let M be a closed minimal surface of a unit m-sphere S^m with Gauss curvature $K \leqq 0$. If there exists a unit normal vector field $\bar{e}$ over M such that $\bar{e}$ is parallel in the normal bundle and the Lipschitz-Killing curvature with respect to $\bar{e}$, $G(p, e)$, is nowhere zero, then M is a Clifford torus in a unit* 3*-sphere $S^3 \subset S^m$.*

THEOREM 2. *Let M be a closed minimal surface of a unit m-sphere with Gauss curvature $K \leqq 0$. If there exists a minimal direction which is parallel in the normal bundle, then M is a Clifford torus in a unit* 3*-sphere $S^3 \subset S^m$.*

From Theorem 2 and the result of Chern-doCarmo-Kobayashi, we obtain

COROLLARY ([1], [4]). *Let M be a closed minimal surface of S^3. If the Gauss curvature of M does not change its sign, then M is either an equatorial sphere or a Clifford torus.*

3. **Proof of Theorem 1.** Suppose that M is a closed minimal surface of a unit m-sphere S^m with Gauss curvature $K \leqq 0$. If there exists a unit normal vector field $\bar{e}$ over M such that $\bar{e}$ is parallel in the normal bundle and the Lipschitz-Killing curvature with respect to $\bar{e}$ is nowhere zero. We consider only the orthonormal frames $(p, e_1, e_2, e_3, \cdots, e_m)$ in B such that $e_m = \bar{e}$ and e_1, e_2 are in the principal directions of e_m. Since M is minimal in S^m, the principal curvatures k_1, k_2 in the direction of e_m are given in the forms:

$$k_1 = h, \quad \text{and} \quad k_2 = -h. \tag{11}$$

Since the Lipschitz-Killing curvature $G(p, e_m) = -h^2 \neq 0$ is defined globally on M, we see that h is defined globally on M. Without loss of generality, we may assume that $h > 0$ on M. Then we have

$$\omega_{1m} = h\omega_1 \quad \text{and} \quad \omega_{2m} = -h\omega_2. \tag{12}$$

By taking exterior derivatives of (12) and applying (5) and (7), we obtain

$$\begin{aligned} 2h\, d\omega_1 + dh \wedge \omega_1 &= \sum \omega_{1r} \wedge \omega_{rm}, \\ 2h\, d\omega_2 + dh \wedge \omega_2 &= -\sum \omega_{2r} \wedge \omega_{rm}. \end{aligned} \tag{13}$$

Since $e_m = \bar{e}$ is parallel in the normal bundle, we have $\omega_{rm} = 0$. Thus (13) reduces to

$$2h\, d\omega_1 + dh \wedge \omega_1 = 0, \qquad 2h\, d\omega_2 + dh \wedge \omega_2 = 0. \tag{14}$$

From (14) we can consider local coordinates (u, v) in an open neighborhood U of a point $p \in M$ such that

$$ds^2 = E\, du^2 + G\, dv^2, \qquad \omega_1 = E^{1/2}\, du, \quad \omega_2 = G^{1/2}\, dv, \tag{15}$$

where ds^2 is the first fundamental form and E and G are local positive functions on U. From (15), equation (14) becomes

$$d(hE) \wedge du = 0, \qquad d(hG) \wedge dv = 0, \tag{16}$$

which shows that (hE) is a function of u, and (hG) is a function of v. By making the following coordinates transformation:

$$u' = \int (hE)^{1/2}\, du, \qquad v' = \int (hG)^{1/2}\, dv, \tag{17}$$

we see, from (15), that there exists a neighborhood V of each point p in M such that there exist isothermal coordinates (u, v) in V such that

$$\begin{aligned} &ds^2 = f\{du^2 + dv^2\}, \qquad \omega_1 = f^{1/2}\, du, \quad \omega_2 = f^{1/2}\, dv, \\ &hf = 1, \end{aligned} \tag{18}$$

where $f = f(u, v)$ is a positive function defined on V. It is well known that the Gauss curvature K is given by

$$K = -(1/2f)\Delta \log (f), \tag{19}$$

with respect to the isothermal coordinates (u, v). Hence, the condition $K \leqq 0$ with $hf = 1$ implies $\Delta \log (h) = -\Delta \log (f) \leqq 0$. From Hopf's lemma, we see that $\log (h)$ is a constant on M. Hence, the Gauss curvature K satisfies $K = (-1/2f)\Delta \log (f) = (h/2)\Delta \log (h) = 0$, identically on M. This implies that M is a closed flat minimal surface in S^m. By a result of Lawson [3], we see that M is, in fact, the Clifford torus in a unit 3-sphere $S^3 \subset S^m$. This completes the proof of the theorem.

4. **Proof of Theorem 2.** Since M is a minimal surface of S^m, we see that the Lipschitz-Killing curvature $G(p, e) \leqq 0$ for all (p, e) in the unit normal bundle. Therefore, if $\bar{e}$ is a minimal direction of M in S^m, then from (9) we obtain

$$G(p, \bar{e}) \leqq -1/(m-2) < 0 \quad \text{on } M. \tag{20}$$

Thus, by Theorem 1 and (20), we obtain Theorem 2.

REMARK. Under the assumption of Theorem 1 or 2, if we replace $K \leqq 0$ by $K \geqq 0$, then we can easily prove that M is either an equatorial sphere or a Clifford torus in $S^3 \subset S^m$, by showing the vanishing of the normal curvature of M in S^m.

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