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MORITA DUALITY FOR ENDOMORPHISM RINGS

ROBERT W. MILLER AND DARRELL R. TURNIDGE

ABSTRACT. A ring R is said to have a left Morita duality with a ring S if there is an additive contravariant equivalence between two categories of left R-modules and right S-modules which include all finitely generated modules in ${}_R\mathfrak{M}$ and $\mathfrak{M}_S$ respectively and which are both closed under submodules and homomorphic images. We show that for such a ring R the endomorphism ring of every finitely generated projective left R-module ${}_RP$ has a left Morita duality with the endomorphism ring of a suitably chosen cofinitely generated injective left R-module ${}_RQ$. Specialized to injective cogenerator rings and quasi-Frobenius rings our results yield results of R. L. Wagoner and Rosenberg and Zelinsky giving conditions when the endomorphism ring of a finitely generated projective left module over an injective cogenerator (quasi-Frobenius) ring is an injective cogenerator (quasi-Frobenius) ring.

1. **Introduction.** A ring R is said to have a *left Morita duality* with a ring S if there is an additive contravariant equivalence between two categories of left R-modules and right S-modules which include all finitely generated modules in ${}_R\mathfrak{M}$ and $\mathfrak{M}_S$ respectively and which are both closed under submodules and homomorphic images. Such a Morita duality exists for R if and only if there exists an injective cogenerator ${}_RU$ for ${}_R\mathfrak{M}$ such that U_S is an injective cogenerator for $\mathfrak{M}_S$ where $S =$ End $({}_RU)$ and such that $R \cong$ End (U_S) via the natural mapping ([**1**], [**3**], [**5**], and [**6**]).

Throughout the following we assume R has such a Morita duality with S via a bimodule ${}_RU_S$. Thus by [**6**] both R and S are semiperfect rings. For $M \in {}_R\mathfrak{M}$ $(N \in \mathfrak{M}_S)$, let $M_S^* = \mathrm{Hom}_R(M, U)$ $({}_RN^* = \mathrm{Hom}_S(N, U))$. Then $(\ )^*$ defines additive contravariant functors from ${}_R\mathfrak{M} \to \mathfrak{M}_S$ and $\mathfrak{M}_S \to {}_R\mathfrak{M}$. A left R-module M (right S-module N) is called *U-reflexive* if ${}_RM \cong {}_RM^{**}$ $(N_S \cong N_S^{**})$ via the natural mappings. The class of U-reflexive modules includes all finitely generated modules in ${}_R\mathfrak{M}$ $(\mathfrak{M}_S)$ and is closed under submodules and homomorphic images.

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We show that for each finitely generated projective left R-module ${}_RP$ the ring $A = \text{End}\,({}_RP)$ has a left Morita duality with the ring $B = \text{End}\,({}_RQ)$ for a suitably chosen cofinitely generated injective left R-module ${}_RQ$. Specialized to injective cogenerator rings and quasi-Frobenius rings our results yield results of R. L. Wagoner [**10**] and Rosenberg and Zelinsky [**7**].

Throughout the following all rings have identity, all modules are unitary and maps are written opposite the scalars.

2. **Results.** A module can be shown to be finitely generated if and only if every ascending chain of proper submodules has proper union. Dually, a module is said to be cofinitely generated if every descending chain of nonzero submodules has nonzero intersection, or equivalently if it has finitely generated essential socle. (See [**9**].)

The following lemma lists several properties of the duality functor $(\ )^*$ which will be required later.

2.1. Lemma. *Let M be a left R-module. Then*

(a) *${}_RM$ is simple if and only if M^*_S is simple.*

(b) *${}_RM$ is finitely generated semisimple if and only if M^*_S is finitely generated semisimple.*

(c) *${}_RM$ is finitely generated if and only if M^*_S is cofinitely generated.*

(d) *${}_RM$ is finitely generated projective if and only if M^*_S is cofinitely generated injective.*

(e) *If ${}_RM$ is reflexive, then* $(\text{Soc}({}_RM))^*_S \cong M^*_S/J(M^*_S)$.

Proof. (a) and (b) are left to the reader.

(c) For ${}_RM$ reflexive and $N \subseteq M$ the map given by $N \to (M/N)^*$ yields a lattice anti-isomorphism between the lattice of submodules of ${}_RM$ and M^*_S.

(d) The "only if" follows by Baer's criteria for injectivity. The "if" follows from the fact that for ${}_RP$ finitely generated, ${}_RP$ is projective if and only if ${}_RP$ is R-projective. (See [**2**].)

(e) The socle of ${}_RM$ is the largest semisimple submodule of ${}_RM$. Hence since ${}_RM$ is reflective we have $(\text{Soc}\,({}_RM))^*_S \cong M^*_S/J(M^*_S)$, the largest semisimple factor module of M^*_S.

Throughout the remainder of this paper we let ${}_RP$ denote a finitely generated projective left R-module and $A = \text{End}\,({}_RP)$. Since R is semiperfect, ${}_RP/J(P)$ is semisimple and contains a copy of each simple image of ${}_RP$. The following notation will be associated with ${}_RP$.

Let $P'_R = \text{Hom}_R\,(P, R)$, ${}_RQ = E({}_RP/J(P))$ (${}_RQ$ is cofinitely generated njective), $B = \text{End}\,({}_RQ)$, and ${}_SQ^{*\prime} = \text{Hom}_S\,(Q^*, S)$.

2.2. PROPOSITION. *Let the notation be as above. Then*
(a) $P_S^* \cong E(Q_S^*/J(Q_S^*))$,
(b) *for* $X \in {}_R\mathfrak{M}$, $\operatorname{Hom}_R(P, X) = 0$ *if and only if* $\operatorname{Hom}_R(X, Q) = 0$,
(c) $A = \operatorname{End}({}_RP) \cong \operatorname{End}(P'_R) \cong \operatorname{End}(P_S^*)$.

PROOF. (a) $\operatorname{Soc}(P_S^*) \cong ({}_RP/J(P))_S^* = (\operatorname{Soc}({}_RQ))_S^* \cong Q_S^*/J(Q_S^*)$ where the isomorphisms follow by (e) of Lemma 2.1.

(b) Let $0 \neq f \in \operatorname{Hom}(P, X)$ and let M be a simple image of $f(P)$ (and hence of P). Then $\operatorname{Hom}(M, Q) \neq 0$ which implies $\operatorname{Hom}(X, Q) \neq 0$ since ${}_RQ$ is injective. Next let $0 \neq f \in \operatorname{Hom}(X, Q)$. Let M be a simple submodule of $f(X)$ (and hence of Q). Then $\operatorname{Hom}(P, M) \neq 0$ which implies $\operatorname{Hom}(P, X) \neq 0$ since P is projective.

(c) $\operatorname{End}(P'_R) \cong \operatorname{End}({}_RP) \cong \operatorname{End}(P_S^*)$ where the second isomorphism is induced by $(\ )^*$ since ${}_RP$ is reflexive.

2.3. LEMMA. *Let the notation be as above. Then* ${}_AP' \otimes {}_RQ$ *is an injective cogenerator for* ${}_A\mathfrak{M}$ *with* $B \cong \operatorname{End}({}_AP' \otimes {}_RQ)$.

PROOF. See Corollary 2 to Theorem 3.2 of [**8**].

2.4. THEOREM. *Let R have a left Morita duality with S via a bimodule* ${}_RU_S$. *Let* ${}_RP$ *be finitely generated projective and let* ${}_RQ = E(P/J(P))$. *Then the ring* $A = \operatorname{End}({}_RP)$ *has a left Morita duality with the ring* $B = \operatorname{End}({}_RQ)$ *via the bimodule* ${}_AP' \otimes {}_RQ_B$.

PROOF. Q_S^* is finitely generated projective. $P_S^* \cong E(Q_S^*/J(Q_S^*))$ is cofinitely generated injective by (d) of Lemma 2.1. So as in Lemma 2.3, $P^* \otimes {}_SQ_B^{*\prime}$ is an injective cogenerator for $\mathfrak{M}_B$ with $A \cong \operatorname{End}(P^* \otimes {}_SQ_B^{*\prime})$. But ${}_AP^* \otimes {}_SQ_B^{*\prime} \cong \operatorname{Hom}_S(Q_S^*, P_S^*) \cong \operatorname{Hom}_R({}_RP, {}_RQ) \cong {}_AP' \otimes {}_RQ_B$. The middle isomorphism follows by $(\ )^*$ since everything in sight is reflexive. Thus ${}_AP' \otimes {}_RQ_B$ yields the required Morita duality.

3. **Applications.** A ring R is called an *injective cogenerator ring* if both ${}_RR$ and R_R are injective cogenerators, i.e. if ${}_RR_R$ yields a Morita duality of R with itself. An injective cogenerator ring which is left (equivalently right) Artinian is called *quasi-Frobenius*.

Our results show that the endomorphism ring of a finitely generated projective right or left R-module has both a left and a right Morita duality if R is an injective cogenerator ring. In general the endomorphism ring of a finitely generated projective right R-module over a quasi-Frobenius ring can fail to be quasi-Frobenius [7].

R. L. Wagoner calls a module ${}_RM$ an *RZ module* if it has the property that every simple homomorphic image of ${}_RM$ is isomorphic to a simple submodule of ${}_RM$. Using the notation of the preceding section one has that for ${}_RP$ a finitely generated projective left R-module with R an injective cogenerator ring, ${}_RP$ is an RZ module if and only if ${}_RQ$ is similar to ${}_RP$. Two modules are said to be *similar* if each is isomorphic to a direct summand of a finite direct sum of copies of the other. Specializing the above theorem to this setting we obtain the following results of R. L. Wagoner [**10**] and Rosenberg and Zelinsky [**7**].

3.1. COROLLARY. *Let R be an injective cogenerator ring. Let ${}_RP$ be a finitely generated projective left RZ module. Then A =* End $({}_RP)$ *is an injective cogenerator ring.*

PROOF. Let ${}_RP$ be finitely generated projective. Since similar modules have Morita equivalent endomorphism rings [**4**] and R is semiperfect we may assume ${}_RP$ is a direct sum of nonisomorphic indecomposable projective (and injective) submodules. Via this reduction if ${}_RP$ is an RZ module ${}_RQ \cong E(P/J(P)) \cong {}_RP$. Thus ${}_AP' \otimes {}_RP_A \cong {}_AA_A$ yields a Morita duality for A with itself. Thus A is an injective cogenerator ring.

3.2. COROLLARY. *Let R be a quasi-Frobenius ring. Let ${}_RP$ be a finitely generated projective left RZ module. Then A =* End $({}_RP)$ *is quasi-Frobenius.*

PROOF. This follows from Corollary 3.1 and the fact that the endomorphism ring of a finitely generated projective left module over an Artinian ring is Artinian.

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DEPARTMENT OF MATHEMATICS, KENT STATE UNIVERSITY, KENT, OHIO 44242