MORITA DUALITY FOR ENDMORPHISM RINGS

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Abstract. A ring $R$ is said to have a left Morita duality with a
ring $S$ if there is an additive contravariant equivalence between two
categories of left $R$-modules and right $S$-modules which include
all finitely generated modules in $\mathcal{M}_R$ and $\mathcal{M}_S$ respectively and which
are both closed under submodules and homomorphic images.

We show that for such a ring $R$ the endomorphism ring of every
finitely generated projective left $R$-module $^P$ has a left Morita
duality with the endomorphism ring of a suitably chosen cofinitely
generated injective left $R$-module $^Q$. Specialized to injective cogenerator rings and quasi-Frobenius rings our results yield results of
R. L. Wagoner and Rosenberg and Zelinsky giving conditions when
the endomorphism ring of a finitely generated projective left module
over an injective cogenerator (quasi-Frobenius) ring is an injective
cogenerator (quasi-Frobenius) ring.

1. Introduction. A ring $R$ is said to have a left Morita duality with
a ring $S$ if there is an additive contravariant equivalence between two
categories of left $R$-modules and right $S$-modules which include all
finitely generated modules in $\mathcal{M}_R$ and $\mathcal{M}_S$ respectively and which are
both closed under submodules and homomorphic images. Such a Morita
duality exists for $R$ if and only if there exists an injective cogenerator
$RU$ for $\mathcal{M}_R$ such that $U_S$ is an injective cogenerator for $\mathcal{M}_S$ where $S = \text{End}(RU)$ and such that $R \cong \text{End}(U_S)$ via the natural mapping ([1],
[3], [5], and [6]).

Throughout the following we assume $R$ has such a Morita duality
with $S$ via a bimodule $RU_S$. Thus by [6] both $R$ and $S$ are semiperfect rings. For $M \in R\mathcal{M}$ ($N \in \mathcal{M}_S$), let $M^*_S = \text{Hom}_R(M, U)$ ($N^*_S = \text{Hom}_S(N, U)$). Then ( )* defines additive contravariant functors from $R\mathcal{M} \to \mathcal{M}_S$ and $\mathcal{M}_S \to R\mathcal{M}$. A left $R$-module $M$ (right $S$-module $N$) is
called $U$-reflexive if $RM \cong R^{M**} (NS \cong N^{**})$ via the natural mappings.
The class of $U$-reflexive modules includes all finitely generated modules in
$R\mathcal{M}$ ($\mathcal{M}_S$) and is closed under submodules and homomorphic images.
We show that for each finitely generated projective left $R$-module $R^P$ the ring $A = \text{End}(R^P)$ has a left Morita duality with the ring $B = \text{End}(R^Q)$ for a suitably chosen cofinitely generated injective left $R$-module $RQ$. Specialized to injective cogenerator rings and quasi-Frobenius rings our results yield results of R. L. Wagoner [10] and Rosenberg and Zelinsky [7].

Throughout the following all rings have identity, all modules are unitary and maps are written opposite the scalars.

2. Results. A module can be shown to be finitely generated if and only if every ascending chain of proper submodules has proper union. Dually, a module is said to be cofinitely generated if every descending chain of nonzero submodules has nonzero intersection, or equivalently if it has finitely generated essential socle. (See [9].)

The following lemma lists several properties of the duality functor $(\ )^\ast$ which will be required later.

2.1. Lemma. Let $M$ be a left $R$-module. Then

(a) $RM$ is simple if and only if $M^\ast_S$ is simple.
(b) $RM$ is finitely generated semisimple if and only if $M^\ast_S$ is finitely generated semisimple.
(c) $RM$ is finitely generated if and only if $M^\ast_S$ is cofinitely generated.
(d) $RM$ is finitely generated projective if and only if $M^\ast_S$ is cofinitely generated injective.
(e) If $RM$ is reflexive, then $(\text{Soc}(RM))^\ast_S \cong M^\ast_S/j(M^\ast_S)$.

Proof. (a) and (b) are left to the reader.
(c) For $R^M$ reflexive and $N \subseteq M$ the map given by $N \rightarrow (M/N)^\ast$ yields a lattice anti-isomorphism between the lattice of submodules of $RM$ and $M^\ast_S$.
(d) The “only if” follows by Baer’s criteria for injectivity. The “if” follows from the fact that for $R^P$ finitely generated, $R^P$ is projective if and only if $R^P$ is $R$-projective. (See [2].)
(e) The socle of $R^M$ is the largest semisimple submodule of $RM$. Hence since $R^M$ is reflective we have $(\text{Soc}(R^M))^\ast_S \cong M^\ast_S/j(M^\ast_S)$, the largest semisimple factor module of $M^\ast_S$.

Throughout the remainder of this paper we let $R^P$ denote a finitely generated projective left $R$-module and $A = \text{End}(R^P)$. Since $R$ is semi-perfect, $R^P/j(R^P)$ is semisimple and contains a copy of each simple image of $R^P$. The following notation will be associated with $R^P$.

Let $P_R = \text{Hom}_R(P, R)$, $RQ = E(R^P/j(R^P))$ ($RQ$ is cofinitely generated injective), $B = \text{End}(RQ)$, and $S\ Q^\ast = \text{Hom}_S(Q^\ast, S)$. 


2.2. Proposition. Let the notation be as above. Then
(a) \( P_S^* \cong E(Q_S^*|J(Q_S^*)) \),
(b) for \( X \in \mathcal{M} \mathcal{R} \), \( \text{Hom}_R (P, X) = 0 \) if and only if \( \text{Hom}_R (X, Q) = 0 \),
(c) \( A = \text{End} (P_R^P) \cong \text{End} (P_R^P) \cong \text{End} (P_S^P) \).

Proof. (a) \( \text{Soc} (P_S^P) \cong (\text{R}P(\text{J}(P)))^* \cong (\text{Soc} (\text{R}Q))_S^* \cong Q_S^*|J(Q_S^*) \) where the isomorphisms follow by (e) of Lemma 2.1.

(b) Let \( 0 \neq f \in \text{Hom} (P, X) \) and let \( M \) be a simple image of \( f(P) \) (and hence of \( P \)). Then \( \text{Hom} (M, Q) \neq 0 \) which implies \( \text{Hom} (X, Q) \neq 0 \) since \( RQ \) is injective. Next let \( 0 \neq f \in \text{Hom} (X, Q) \). Let \( M \) be a simple submodule of \( f(X) \) (and hence of \( Q \)). Then \( \text{Hom} (P, M) \neq 0 \) which implies \( \text{Hom} (P, X) \neq 0 \) since \( P \) is projective.

(c) \( \text{End} (P_R^P) \cong \text{End} (P_R^P) \cong \text{End} (P_S^P) \) where the second isomorphism is induced by ( ) * since \( R_P \) is reflexive.

2.3. Lemma. Let the notation be as above. Then \( A \mathcal{P}' \otimes RQ \) is an injective cogenerator for \( \mathcal{M} \mathcal{W} \) with \( B \cong \text{End} (A \mathcal{P}' \otimes RQ) \).

Proof. See Corollary 2 to Theorem 3.2 of [8].

2.4. Theorem. Let \( R \) have a left Morita duality with \( S \) via a bimodule \( R_US \). Let \( P_R \) be finitely generated projective and let \( \text{R}Q = E(P_J(P)) \). Then the ring \( A = \text{End} (\text{R}P) \) has a left Morita duality with the ring \( B = \text{End} (\text{R}Q) \) via the bimodule \( A \mathcal{P}' \otimes RQ_B \).

Proof. \( Q_S^* \) is finitely generated projective. \( P_S^* \cong E(Q_S^*|J(Q_S^*)) \) is cofinitely generated injective by (d) of Lemma 2.1. So as in Lemma 2.3, \( P_S^* \otimes S_S^* Q_B^* \) is an injective cogenerator for \( \mathcal{M}_B \) with \( A \cong \text{End} (P_S^* \otimes S_S^* Q_B^*) \). But \( A \mathcal{P}' \otimes S_S^* Q_B^* \cong \text{Hom}_S (Q_S^*, P_S^*) \cong \text{Hom}_R (P_R, RQ) \cong A \mathcal{P}' \otimes RQ_B \). The middle isomorphism follows by ( ) * since everything in sight is reflexive. Thus \( A \mathcal{P}' \otimes RQ_B \) yields the required Morita duality.

3. Applications. A ring \( R \) is called an injective cogenerator ring if both \( R_R \) and \( R_R \) are injective cogenerators, i.e. if \( R_R R_R \) yields a Morita duality of \( R \) with itself. An injective cogenerator ring which is left (equivalently right) Artinian is called quasi-Frobenius.

Our results show that the endomorphism ring of a finitely generated projective right or left \( P \)-module has both a left and a right Morita duality if \( R \) is an injective cogenerator ring. In general the endomorphism ring of a finitely generated projective right \( R \)-module over a quasi-Frobenius ring can fail to be quasi-Frobenius [7].

R. L. Wagoner calls a module \( R_M \) an \( RZ \) module if it has the property that every simple homomorphic image of \( R_M \) is isomorphic to a simple submodule of \( R_M \). Using the notation of the preceding section one has that for \( R_P \) a finitely generated projective left \( R \)-module with \( R \) an injective
cogenerator ring, _R_ is an _RZ_ module if and only if _RQ_ is similar to _RP_.

Two modules are said to be similar if each is isomorphic to a direct summand of a finite direct sum of copies of the other. Specializing the above theorem to this setting we obtain the following results of R. L. Wagoner [10] and Rosenberg and Zelinsky [7].

3.1. Corollary. Let _R_ be an injective cogenerator ring. Let _RP_ be a finitely generated projective left _RZ_ module. Then _A_ = End (_RP_) is an injective cogenerator ring.

Proof. Let _RP_ be finitely generated projective. Since similar modules have Morita equivalent endomorphism rings [4] and _R_ is semiperfect we may assume _RP_ is a direct sum of nonisomorphic indecomposable projective (and injective) submodules. Via this reduction if _RP_ is an _RZ_ module _RQ_ ^ E(P) _RP_. Thus _A_ ^ _RP_ ^ yields a Morita duality for _A_ with itself. Thus _A_ is an injective cogenerator ring.

3.2. Corollary. Let _R_ be a quasi-Frobenius ring. Let _RP_ be a finitely generated projective left _RZ_ module. Then _A_ = End (_RP_) is quasi-Frobenius.

Proof. This follows from Corollary 3.1 and the fact that the endomorphism ring of a finitely generated projective left module over an Artinian ring is Artinian.

References


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