PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 1, January 1972

A NONPRINCIPAL INVARIANT SUBSPACE OF THE HARDY SPACE ON THE TORUS

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ABSTRACT. Let $H^2(U^n)$ be the usual Hardy space (with index 2) of holomorphic functions on U^n, the unit polydisc in complex n-space. A subspace of $H^2(U^n)$ is *invariant* if closed under multiplication by the coordinate functions. To solve a problem left open in a paper of P. R. Ahern and D. N. Clark and a book by W. Rudin the author constructs a closed invariant subspace M of $H^2(U^2)$ with (1) an f in M never vanishing on U^2 and (2) each g in M being contained in a proper closed invariant subspace of M. This easily extends to $n \geqq 2$.

We answer a question posed in Ahern and Clark's paper *Invariant subspaces and analytic continuation in several variables*, p. 967 and Rudin's book *Function theory in polydiscs*, p. 78.

Let T^2 be the torus with its dual Z^2 realized as the integral lattice points of the plane under the usual pairing $\langle (m, n), (e^{ix}, e^{iy})\rangle = e^{mix}e^{niy}$. For any subset A of Z^2 let $H^2(A)$ be the subspace of $L^2(T^2)$ consisting of functions whose Fourier coefficients vanish off A. If A is taken to be the set $S = \{(m, n): m \geqq 0, n \geqq 0\}$, then $H^2(S)$ is the Hardy space on the torus. Let U^2 be the set of pairs (z, w) with z and w complex and $|z| < 1$, $|w| < 1$. To each function f in $H^2(S)$ with Fourier series $\sum_{m\geqq 0, n\geqq 0} a_{mn}e^{mix}e^{niy}$ we associate the function $\sum_{m\geqq 0, n\geqq 0} a_{mn}z^m w^n$, analytic on U^2, which we also denote by f.

If A is a subset of Z^2 and M is a closed linear subspace of $L^2(T^2)$, M is *A-invariant* whenever f in M and c in A implies the function cf also lies in M. The question we answer is the following: Suppose M is an S-invariant subspace of the Hardy space $H^2(S)$ and that for each point

Presented to the Society, January 23, 1971 under the title *A nonprincipal invariant subspace*; received by the editors February 5, 1971.

AMS 1970 *subject classifications*. Primary 32A30, 46E20, 47D05; Secondary 32A99, 42A92.

Key words and phrases. Hardy space, invariant subspace, outer function, multiple Fourier series, half-plane, semigroup of characters.

[1] This example is contained in the author's Ph.D. thesis, written under the direction of Professor H. Helson at the University of California, Berkeley. The author was partially supported by a National Science Foundation Graduate Fellowship.

(z, w) in U^2 there is an f in M (which may vary with (z, w)) such that $f(z, w) \neq 0$. Does it follow that there is a single function h such that M is the smallest S-invariant subspace containing h? If such an h exists we will say that M is *principal* and that h is *cyclic* for M.

The answer is no. Before turning to our counterexample, we include some preparatory remarks. If the subset A of Z^2 is a semigroup with $A \cup (-A) = Z^2$ and $A \cap (-A) = \{(0, 0)\}$, A is called a *half-plane*. If h lies in $H^2(A)$ and the smallest A-invariant subspace containing h is precisely $H^2(A)$, h is called *A-outer*. For a semigroup A, $H^2(A)$ is itself A-invariant and any nonzero constant function is A-outer. The classical Hardy space is associated with the unit circle, and consists of square integrable functions with Fourier coefficients vanishing off the non-negative integers. Such a function $\sum_{m \geqq 0} a_m e^{mix}$ also has an extension to an analytic function $\sum_{m \geqq 0} a_m z^m$ on the open unit disc.

To construct our counterexample we choose q to be a nonconstant, singular inner function in the classical Hardy space of the unit disc [**2**, p. 67]. This means, first, that q never vanishes inside the unit circle, and, second, that q has modulus one almost everywhere on the unit circle. Now we let M be the S-invariant subspace of $H^2(S)$ generated by f and g, where $f(e^{ix}, e^{iy}) = q(e^{ix})$ and $g(e^{ix}, e^{iy}) = e^{iy}$. M meets the requirements for $f(z, w) \neq 0$ for every (z, w) in U^2, and we will show that M is not principal.

Suppose that M were principal. Let h be cyclic for M and let M_1 be the S_1-invariant subspace generated by h under S_1, the half-plane $\{(m, n): m \geqq 0$, and $m = 0$ implies that $n \geqq 0\}$. If q is $\sum_{m \geqq 0} a_m e^{mix}$ we see that

$$\begin{aligned} a_0 &= \sum_{m \geqq 0} a_m e^{mix} - \sum_{m \geqq 1} a_m (e^{mix} e^{-iy}) e^{iy} \\ &= f - \sum_{m \geqq 1} a_m (e^{mix} e^{-iy}) g \end{aligned}$$

lies in M_1 because $c_m = (m, -1)$ lies in S_1 for $m \geqq 1$, and so $c_m g = e^{mix} e^{-iy} g$ lies in M_1. But $a_0 = f(0, 0) = q(0) \neq 0$, so that the constant functions lie in M_1. Thus h is S_1-outer.

This property of h for any half-plane containing the support of the Fourier transform of h is equivalent to an analytic condition independent of the half-plane. Indeed, if A is a half-plane with h in $H^2(A)$, then h is A-outer if and only if

$$\int \log |h(e^{ix}, e^{iy})| \, dx \, dy = \log \left| \int h(e^{ix}, e^{iy}) \, dx \, dy \right| > -\infty,$$

[**3**, p. 212]. In particular h is also S_2-outer for $S_2 = \{(m, n): n \geqq 0$, and $n = 0$ implies $m \geqq 0\}$.

Viewing the situation on the nonnegative integral points of the horizontal axis, one sees that q must be an outer function for the classical Hardy space. Indeed, let $B = \{(m, 0): m \geqq 0\}$ and let P be the orthogonal projection onto $H^2(B)$. Now the S-invariant subspaces generated by $\{f, g\}$ and $\{h\}$ are the same. Since $S_2 \supset S$, the S_2-invariant subspaces generated by $\{f, g\}$ (denoted by $M_2(f, g)$) and generated by $\{h\}$ (denoted by $M_2(h)$) are the same. But $P[M_2(f, g)]$ is the closed linear span of all $e^{mix}f$, for $m \geqq 0$, while $P[M_2(h)] = P[H^2(S_2)] = H^2(B)$. Thus, by the definition of f from q, the latter must be outer in the classical Hardy space. But this is known to be false [**2**, p. 63]. This contradiction means that our assumption that M be principal was incorrect.

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