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ON A FACTORIZATION THEOREM OF D. LOWDENSLAGER[1]

V. MANDREKAR AND H. SALEHI

ABSTRACT. For a positive-definite infinite-dimensional matrix-valued function M defined on the unit circle a factorization theorem for M in the form $M = AA^*$, where A is a function with Fourier series $\sum_{n>0} A_n e^{in\theta}$, is proved. The theorem, as was originally stated by D. Lowdenslager, contained an error. Based on our study concerning the completeness of the space of square-integrable operator-valued functions (not necessarily bounded) with respect to a nonnegative bounded operator-valued measure a correct proof of the factorization problem is provided. This work subsumes several known results concerning the factorization problem.

1. **Introduction.** For a positive-definite infinite-dimensional matrix-valued function M defined on the unit circle, D. Lowdenslager [**6**] gave a proof of factorization theorem for M in the form $M = AA^*$, where A is a function with Fourier series $\sum_{n>0} A_n e^{in\theta}$. Theorem 2 of [**6**] as stated contains an error as shown by the counterexample in [**1**]. In case M and M^{-1} are bounded, Theorem 2 of [**6**] remains valid (Helson [**3**, p. 118]). In [**8**], M. Nadkarni remarks that the completeness of the space $L_{2,M}$ presumed in [**6**] is probably not valid without additional assumptions. Recently, we have studied, in [**7**], the completeness of the space $L_{2,M}$ of square-integrable operator-valued functions (not necessarily bounded) with respect to a nonnegative bounded operator-valued measure M. Based on our study of $L_{2,M}$, we state a correct form of Theorem 2 of [**6**] such that Lowdenslager's proof is corrected. Also, this provides a form of factorization theorem more general than the one in [**3**] and [**1**]. In fact, as a consequence of our theorem (cf. Theorem 3.3), the main theorem of Douglas [**1**, Theorem 1] can be deduced.

We remark that the proof of Lowdenslager is in the same spirit as the original work of H. Wold [**9**] on prediction theory.

2. **Preliminaries.** For any two separable (complex) Hilbert-spaces $\mathcal{H}$, $\mathcal{K}$ with inner products $(\cdot, \cdot)_{\mathcal{H}}$, $(\cdot, \cdot)_{\mathcal{K}}$ and norms $|\ |_{\mathcal{H}}$, $|\ |_{\mathcal{K}}$ we denote

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by

(a) $O(\mathcal{H}, \mathcal{K})$, the class of all linear operators from $\mathcal{H}$ into $\mathcal{K}$;
(b) $B(\mathcal{H}, \mathcal{K})$, the class of all bounded operators on $\mathcal{H}$ into $\mathcal{K}$;
(c) $B^+(\mathcal{H}, \mathcal{H})$, the class of all nonnegative definite operators in $B(\mathcal{H}, \mathcal{H})$;
(d) $HS(\mathcal{H}, \mathcal{K})$, the class of all Hilbert-Schmidt operators in $B(\mathcal{H}, \mathcal{K})$.
(e) $C(\mathcal{H}, \mathcal{K})$, the class of compact operators in $B(\mathcal{H}, \mathcal{K})$.

Let $\mathcal{B}$ be a σ-ring of subsets of a space Ω. We shall need a concept of measurability of $O(\mathcal{H}, \mathcal{K})$-valued functions. For $B(\mathcal{H}, \mathcal{K})$-valued functions the notion of measurability (weak and strong) has been defined before [5, p. 74]. In view of the separability of $\mathcal{H}$ and $\mathcal{K}$ the weak and strong measurability are equivalent. We shall refer to both notions as measurability. Following is our extension of the notion of measurability.

2.1 DEFINITION. Let Φ be an $O(\mathcal{H}, \mathcal{K})$-valued function on Ω; then Φ is said to be $\mathcal{B}$-measurable if there exists a sequence of $\mathcal{B}$-measurable $B(\mathcal{H}, \mathcal{K})$-valued functions (Φ_n) such that for each $\omega \in \Omega$ and for each x in the domain of $\Phi(\omega)$, we have $\lim_{n\to\infty} |\Phi_n(\omega)x - \Phi(\omega)x|_{\mathcal{K}} = 0$.

Let μ be measurable on $\mathcal{B}$ and M be a $\mathcal{B}$-measurable $B^+(\mathcal{H}, \mathcal{H})$-valued function on Ω such that for each $x \in \mathcal{H}$, $\int_\Omega |Mx|_{\mathcal{H}}\, d\mu$ is finite. For the countably additive $B^+(\mathcal{H}, \mathcal{H})$-valued measure given by indefinite integral of M, we define $L_{2,M}$ as follows [7, Remark 4.11].

2.2 DEFINITION. The space $L_{2,M}$ consists of $\mathcal{B}$-measurable $O(\mathcal{H}, \mathcal{K})$-valued functions Φ satisfying (i) $\Phi\sqrt{M}$ is $HS(\mathcal{H}, \mathcal{K})$-valued a.e. $[\mu]$ and (ii) the real-valued function $|\Phi\sqrt{M}|_E$ [2] is square-integrable μ.

In $L_{2,M}$, $\Phi = \Psi$ iff $\Phi\sqrt{M} = \Psi\sqrt{M}$ a.e. $[\mu]$. If M is $C(\mathcal{H}, \mathcal{K})$-valued we showed [7, Theorem 4.19] that $L_{2,M}$ is a (complete) Hilbert space. In the proof of the completeness of $L_{2,M}$, we made use of the fact that M^-, the generalized inverse[3] of a $\mathcal{B}$-measurable compact operator valued function M is $\mathcal{B}$-measurable. As we mentioned in Remark 4.16 in [7], $L_{2,M}$ remains complete if we know that M^- is measurable. We establish this result below.

2.3 LEMMA. *Let A be a $\mathcal{B}$-measurable $B^+(\mathcal{H}, \mathcal{H})$-valued function then A^- is a $\mathcal{B}$-measurable $O(\mathcal{H}, \mathcal{H})$-valued function.*

PROOF. Consider for each a, b $(-\infty < a < b < +\infty)$, the function $I_{(a,b]}(x) = 1$ for $x \in (a, b]$ and $= 0$ for $x \notin (a, b]$ defined on R. Then there exists a sequence $\{p_n(\cdot)\}$ of polynomials such that $|p_n(x)| \leqq 1$ for all $x \in R$ and $\lim_{n\to\infty} p_n(x) = I_{(a,b]}(x)$ for each x. Consider now $A(\omega) = \int \lambda E_\omega(d\lambda)$. Then, for each $h \in \mathcal{H}$,

$$\|E_\omega(a, b]h - p_n(A(\omega))h\|_{\mathcal{H}}^2 = \int |I_{(a,b]}(x) - p_n(x)|^2\,(E_\omega(dx)h, h).$$

[2] $|\ |_E$ denotes the Hilbert-Schmidt norm.

[3] For the definition of generalized inverse see [7, Definition 2.11].

Hence by the Lebesgue dominated convergence theorem, as $n \to \infty$,

$$\|E_\omega(a, b]h - p_n(A(\omega))h\|^2_{\mathcal{H}} \to 0.$$

This implies $E_\omega(a, b]$ is a measurable function of ω. Hence

$$\int_{\lambda > 1/n} \lambda^- E_\omega(d\lambda)$$

is a measurable function of ω. Now by Hestenes [**4**, pp. 1326 and 1337] we get

$$\left\| \dot{A}^-(\omega)h - \int_{\lambda > 1/n} \lambda^- E_\omega(d\lambda)h \right\|_{\mathcal{H}} \to 0$$

for each $h \in$ domain of $A^-(\omega)$. Therefore $A^-(\cdot)$ is $\mathcal{B}$-measurable in the sense of Definition 2.1.

3. **Factorization problem.** Let M be a measurable $B^+(\mathcal{H}, \mathcal{H})$-valued function defined on the unit circle which is Bochner integrable with respect to Lebesgue measure μ.

3.1 DEFINITION. We say that M is factorable if there exists a measurable $B(\mathcal{H}, \mathcal{H})$-valued function A such that

(i) $M = AA^*$,

(ii) $A(e^{i\theta}) = \sum_{n \geq 0} A_n e^{in\theta}$,

where $A_n \in B(\mathcal{H}, \mathcal{H})$ and the convergence is taken in the strong sense.

3.2 THEOREM. *Let M_1, M_2 be measurable $B^+(\mathcal{H}, \mathcal{H})$-valued functions defined on the unit circle which are Bochner integrable with respect to Lebesgue measure μ. If* (i) $M_2 \geqq M_1$,[4] (ii) *the injection map is one-one on L_{2,M_2} into L_{2,M_1}. Then M_1 factorable implies M_2 is factorable.*

Under the above assumptions the proof in [**6**] is valid and hence is omitted. In case of [**6**] the error comes from the fact that the injection map is not one-one.

3.3 COROLLARY (HELSON [**3**, p. 118]). *Let M_1 and M_2 be $B^+(\mathcal{H}, \mathcal{H})$-valued functions such that $M_2 \geqq M_1$. Suppose that M_1^{-1} exists and is bounded and M_1 is factorable then M_2 is factorable.*

PROOF. Since M_1^{-1} is bounded, the space L_{2,M_1} consists only of $HS(\mathcal{H}, \mathcal{K})$-valued functions [**7**, Remark 4.16]. Further, $M_2 \geqq M_1$ and $\mathcal{N}_{M_1}$[5] $= \{0\}$ implies $\mathcal{N}_{M_2} = \{0\}$. This implies that the injection map is one-one.

[4] This implies that $L_{2,M_2} \subseteq L_{2,M_1}$ and that the injection map is a contraction (see [**7**, Definition 2.2] and [**2**, Theorem 1]).

[5] For an operator A, $\mathcal{N}_A$ denotes the null of A and $\mathcal{R}_A$ will denote the range of A.

3.4 LEMMA. *Let M_1, M_2 be measurable $B^+(\mathcal{H}, \mathcal{H})$-valued functions defined on the unit circle and Bochner integrable with respect to Lebesgue measure μ and satisfying*

(i) $M_2 \geqq M_1$,

(ii) $\mathcal{N}_{M_1} = \mathcal{N}_{M_2}$,

(iii) *there exists a measurable positive scalar-valued function such that* $\sqrt{M_2} M_1 \sqrt{M_2} \leqq \varphi M_1$.

Then the injection map is one-one.

PROOF. By (iii) and Theorem 1 of [2], $\mathcal{R}(\sqrt{M_1}) \supseteq \mathcal{R}(\sqrt{M_2}\sqrt{M_1})$. Now $\|\Phi\|_{M_1} = 0$ implies $\Phi\sqrt{M_1} = 0$ a.e. $[\mu]$. Hence $\|\Phi\|_{M_1} = 0$ implies $\Phi\sqrt{M_2}\sqrt{M_1} = 0$ a.e. $[\mu]$. But $\Phi \in L_{2,M_2}$, $\Phi\sqrt{M_2}$ is bounded giving $\Phi\sqrt{M_2} = 0$ on closure of $\mathcal{R}(\sqrt{M_1})$. This and (ii) implies that $\Phi\sqrt{M_2} = 0$ a.e. μ giving $\|\Phi\|_{M_2} = 0$.

Combining Theorem 3.2 and Lemma 3.4, we obtain:

3.5 COROLLARY [1, THEOREM 1]. *Let M_2, M_1 satisfy the hypotheses of Lemma* 3.4. *If M_1 is factorable then so is M_2.*

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DEPARTMENT OF STATISTICS AND PROBABILITY, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823

Current address (Salehi): Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706