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A STRONG HOMOTOPY AXIOM FOR ALEXANDER COHOMOLOGY

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ABSTRACT. It is shown that the following form of the homotopy axiom holds for Alexander-Čech cohomology. Suppose that X and Y are any spaces, that T is a compact, connected space, and that G is an abelian group which either admits the structure of a compact topological group or is the additive group of a finite-dimensional vector space. Then for any continuous function $F: X \times T \to Y$, one has $F_r^* = F_s^*: H^*(Y; G) \to H^*(X; G)$ for all $r, s \in T$, where $F_t: X \to Y$ is defined by $F_t(x) = F(x, t)$.

1. **Introduction.** The purpose of this paper is to prove the following theorem.

THEOREM. *Suppose that X and Y are any spaces, that T is a compact, connected space, and that G is a compact abelian topological group or a finite-dimensional vector space. Then for any continuous function $F: X \times T \to Y$, one has*

$$F_r^* = F_s^*: H^*(Y; G) \to H^*(X; G)$$

for all r, s in T, where $F_t: X \to Y$ is defined by $F_t(x) = F(x, t)$.

While we prove the theorem for Alexander (more accurately, Alexander-Kolmogoroff) cohomology, it also holds for Čech cohomology since the two are naturally equivalent [2]. The theorem also holds for topological pairs, as a straightforward modification of our proof will show. We restrict our attention to the single space case for simplicity of presentation.

In case T is an arc, the theorem is well known (no restriction on G), following either from its truth in the Čech theory [1] and the equivalence of the Alexander and Čech theories [2] or from a direct proof for the Alexander theory ([6] or [5, p. 314]). On the other hand, it is known that if X is compact, the theorem holds with T any connected space (again, no rectriction on G) as an easy generalization of the proof in [4, p, 416] shows. The freedom of choosing the parameter space to be continuum, which is allowed by taking X to be compact, has proved very useful in the study

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of topological semigroups. Our theorem retains this freedom of choice of parameter space while removing the compactness restriction on X, provided one chooses an appropriate coefficient group. It is a trivial observation, of course, that one does not really need to assume T to be compact, only that each pair of points of T is contained in some compact, connected subset of T (e.g., T locally compact, locally connected, connected). For further discussion of the problem the reader is referred to the review by K. Hofmann in Mathematical Reviews **32** (1966), p. 514.

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2. **Preliminaries.** In this section we collect several preliminary results we will need later.

If X is a set and $\mathcal{U}$ is a collection of sets, we denote by $X_{\mathcal{U}}$ the simplicial complex whose simplexes are the finite nonempty subsets of X which are contained in some member of $\mathcal{U}$.

PROPOSITION 1. *For each space X and open cover $\mathcal{U}$ of X there is a homomorphism $i_{\mathcal{U}}: H^*(X_{\mathcal{U}}) \to H^*(X)$ such that $\{H^*(X), i_{\mathcal{U}}\}$ is a direct limit of the direct system $\{H^*(X_{\mathcal{U}}), i^*_{\mathcal{U},\mathcal{U}'}, \operatorname{cov} X\}$ where $i_{\mathcal{U},\mathcal{U}'} \cdot X_{\mathcal{U}} \subset X_{\mathcal{U}'}$ ($\mathcal{U}$ refines $\mathcal{U}'$). Furthermore, if $f: X \to Y$ is a map and if $\mathcal{U}$ and $\mathcal{V}$ are covers of X and Y, respectively, such that $\{f(U) \mid U \in \mathcal{U}\}$ refines $\mathcal{V}$, then the diagram*

$$\begin{array}{ccc} H^*(Y) & \xrightarrow{f^*} & H^*(X) \\ \uparrow{\scriptstyle i_{\mathcal{V}}} & & \uparrow{\scriptstyle i_{\mathcal{U}}} \\ H^*(Y_{\mathcal{V}}) & \xrightarrow{f_0^*} & H^*(X_{\mathcal{U}}) \end{array}$$

commutes, where $f_0: X_{\mathcal{U}} \to Y_{\mathcal{V}}$ is the simplicial map induced by f.

For a proof of this proposition, see [**5**, p. 312].

Although the functor H^* does not, in general, commute with inverse limits, the following proposition holds. Its proof is an appropriate modification of that given for the special case of exact sequences in [**3**, p. 226].

PROPOSITION 2. *On either the category of cochain complexes over compact abelian topological groups or the category of cochain complexes over finite-dimensional vector spaces over a fixed field, the functor H^* commutes with inverse limits.*

For a simplicial complex K, $C^p(K; G)$ is defined as $\{\varphi \mid \varphi: K_p \to G\}$, where $K_p = \{(\chi_0, \cdots, \chi_p) \mid \{\chi_0, \cdots, \chi_p\} \in K\}$. $C^p(K; G)$ is then just a cartesian product of copies of G. Hence, we have the following propositions. The proof of the continuity, resp. linearity, of δ and $f^{\#}$ is straightforward.

PROPOSITION 3. *If G is a compact abelian topological group and K is a simplicial complex, then $C^p(K; G)$ is a compact abelian topological group. Furthermore, $\delta: C^p(K; G) \to C^{p+1}(K; G)$ and, for a simplicial map $f: K \to K', f^{\#}: C^p(K'; G) \to C^p(K; G)$ are continuous.*

PROPOSITION 3′. *If G is a finite-dimensional vector space and K is a finite simplicial complex, then $C^p(K; G)$ is a finite-dimensional vector space. Furthermore, $\delta: C^p(K; G) \to C^{p+1}(K; G)$ and, for a simplicial map $f: K \to K', f^{\#}: C^p(K'; G) \to C^p(K; G)$ are linear transformations.*

The proof of the following proposition is simple and hence omitted.

PROPOSITION 4. *Suppose K is a simplicial complex and $\mathcal{C}$ is a collection of subcomplexes of K satisfying*

(i) $\bigcup \mathcal{C} = K$, *and*

(ii) *if $A, B \in \mathcal{C}$, then $A \cup B$ is contained in some member of $\mathcal{C}$.*

Then, for any abelian group G, $\{C^*(K; G), p_A^{\#}\}$ is an inverse limit of the inverse system $\{C^*(A; G), i_{A,B}^{\#}, \mathcal{C}\}$, where $i_{A,B}: A \subset B$ and $p_A: A \subset K$.

3. **Proof of the Theorem.** We prove the following Lemma, which is easily seen to be equivalent to the theorem.

LEMMA. *Suppose that X and T are spaces with T compact and connected and that G is either a compact abelian topological group or a finite-dimensional vector space. If $\lambda_t: X \to X \times T$ is defined for each t in T by $\lambda_t(x) = (x, t)$, then $\lambda_r^* = \lambda_s^*: H^*(X \times T; G) \to H^*(X; G)$ for all r, s in T.*

PROOF. Let $g \in H^p(X \times T)$. We wish to show that $\lambda_r^*(g) = \lambda_s^*(g)$. From Proposition 1 we have that there is an open cover $\mathcal{V}$ of $X \times T$ and $g_{\mathcal{V}} \in H^p((H \times T))_{\mathcal{V}}$ such that $i_{\mathcal{V}}(g_{\mathcal{V}}) = g$, where $i_{\mathcal{V}}$ is the canonical injection. Using the compactness of T, we get for each $x \in X$ an open set U_x about x and an open cover $\mathcal{D}_{U_x}$ of T such that $\{U_x \times D \mid D \in \mathcal{D}_{U_x}\}$ refines $\mathcal{V}$. We let $\mathcal{U}$ denote the open cover $\{U_x \mid x \in X\}$ of X and let $\mathcal{K}$ denote the collection of all ordered pairs $(\mathcal{G}, F)$ where $\mathcal{G}$ is a finite subcollection of $\mathcal{U}$ and F is a finite subset of X with $F \subset \bigcup \mathcal{G}$. By defining $(\mathcal{G}_1, F_1) \leqq (\mathcal{G}_2, F_2)$ iff $\mathcal{G}_1 \subset \mathcal{G}_2$ and $F_1 \subset F_2$, $\mathcal{K}$ becomes a directed set. It is clear that, for each t in T, λ_t induces simplicial maps $\lambda_t^{\mathcal{U}}: X_{\mathcal{U}} \to (X \times T)_{\mathcal{V}}$ and $\lambda_t^K: F_{\mathcal{G}} \to (X \times T)_{\mathcal{V}}$, $K = (\mathcal{G}, F) \in \mathcal{K}$.

We show first that, for each $K = (\mathcal{G}, F) \in \mathcal{K}$, the chain maps $(\lambda_r^K)^{\#}$, $(\lambda_s^K)^{\#}: C^*((X \times T)_{\mathcal{V}}) \to C^*(F_{\mathcal{G}})$ are chain homotopic. Let $K = (\mathcal{G}, F) \in \mathcal{K}$

with $\mathcal{C} = \{U_1, \cdots, U_n\}$ and let $\mathfrak{D}$ be a common open refinement of $\mathfrak{D}_{U_1}, \cdots, \mathfrak{D}_{U_n}$ covering T. Since T is connected there exist $D_1, \cdots, D_k \in \mathfrak{D}$ and $p_0, \cdots, p_k$ such that $r = p_0 \in D_1$, $s = p_k \in D_k$ and $p_i \in D_i \cap D_{i+1}$, $1 \leqq i \leqq k-1$. For each $i = 1, \cdots, k$, $\lambda^K_{p_i}$ and $\lambda^K_{p_i - 1}\colon F_{\mathcal{C}} \to (X \times T)_{\mathfrak{V}}$ are contiguous since, for each $i = 1, \cdots, n$, $\lambda^K_{p_i}(U_i) = U_i \times \{p_i\} \subset U_i \times D_i$ and $\lambda^K_{p_i - 1}(U_i) = U_i \times \{p_{i-1}\} \subset U_i \times D_i$ with $U_i \times D_i \subset U_i \times D \subset V$ for some $D \in \mathfrak{D}_{U_i}$ and $V \in \mathfrak{V}$. It follows that, for each $i = 1, \cdots, k$, $(\lambda^K_{p_i})^{\#}$ and $(\lambda^K_{p_i - 1})^{\#}$ are chain homotopic and hence so are $(\lambda^K_r)^{\#}$ and $(\lambda^K_s)^{\#}$.

We now complete the proof of the Lemma. It is immediate from Proposition 4 that $\{C^*(X_{\mathfrak{U}}), i^{\#}_K\}$ is an inverse limit of the inverse system $\{C^*(F_{\mathcal{C}}), i^{\#}_{K,K'}, \mathfrak{K}\}$ where $i_K\colon F_{\mathcal{C}} \subset X_{\mathfrak{U}}$ and $i_{K,K'}\colon F_{\mathcal{C}} \subset F'_{\mathcal{C}'}$, $K \leqq K'$, $K = (\mathcal{C}, F)$, $K' = (\mathcal{C}', F')$. From Propositions 2, 3, and 3′ we then have that $\{H^p(X_{\mathfrak{U}}), i^*_K\}$ is an inverse limit of the system $\{H^p(F_{\mathcal{C}}), i^*_{K,K'}, \mathfrak{K}\}$. For each $t \in T$ and $K, K' \in \mathfrak{K}$, $K \leqq K'$, the diagrams

$$\begin{array}{ccccc} & & H^p((X \times T)_{\mathfrak{V}}) & & \\ & {\scriptstyle (\lambda^{K'}_t)^*}\swarrow & & \searrow{\scriptstyle (\lambda^K_t)^*} & \\ H^p(F'_{\mathcal{C}'}) & & \xrightarrow{\quad i^*_{K,K'} \quad} & & H^p(F_{\mathcal{C}}) \end{array}$$

and

$$\begin{array}{ccc} H^p((X \times T)_{\mathfrak{V}}) & \xrightarrow{(\lambda^{\mathfrak{U}}_t)^*} & H^p(X_{\mathfrak{U}}) \\ & {\scriptstyle (\lambda^K_t)^*}\searrow & \big\downarrow {\scriptstyle i^*_K} \\ & & H^p(F_{\mathcal{C}}) \end{array}$$

commute. But $(\lambda^K_r)^* = (\lambda^K_s)^*$ for each $K \in \mathfrak{K}$ since $(\lambda^K_r)^{\#}$ and $(\lambda^K_s)^{\#}$ are chain homotopic, so from the uniqueness condition in the characteristic property of inverse limits one has $(\lambda^{\mathfrak{U}}_r)^* = (\lambda^{\mathfrak{U}}_s)^*$. From this and the commutativity of

$$\begin{array}{ccc} H^p(X \times T) & \xrightarrow{\lambda^*_t} & H^p(X) \\ {\scriptstyle i_{\mathfrak{V}}}\big\uparrow & & \big\uparrow{\scriptstyle i_{\mathfrak{U}}} \\ H^p((X \times T)_{\mathfrak{V}}) & \xrightarrow{(\lambda^{\mathfrak{U}}_t)^*} & H^p(X_{\mathfrak{U}}) \end{array}$$

(Proposition 1) one obtains $\lambda^*_r(g) = \lambda^*_s(g)$, completing the proof.

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