A STRONG HOMOTOPY AXIOM FOR ALEXANDER
COHOMOLOGY

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Abstract. It is shown that the following form of the homotopy
axiom holds for Alexander-Čech cohomology. Suppose that X and
Y are any spaces, that T is a compact, connected space, and that G
is an abelian group which either admits the structure of a compact
topological group or is the additive group of a finite-dimensional
vector space. Then for any continuous function $F: X \times T \to Y$, one
has $F^*_r = F^*_s: H^*(Y; G) \to H^*(X; G)$ for all $r, s \in T$, where $F_t: X \to Y$ is defined by $F_t(x) = F(x, t)$.

1. Introduction. The purpose of this paper is to prove the following
theorem.

Theorem. Suppose that X and Y are any spaces, that T is a compact,
connected space, and that G is a compact abelian topological group or a
finite-dimensional vector space. Then for any continuous function $F: X \times T \to Y$, one has

$$F^*_r = F^*_s: H^*(Y; G) \to H^*(X; G)$$

for all $r, s \in T$, where $F_t: X \to Y$ is defined by $F_t(x) = F(x, t)$.

While we prove the theorem for Alexander (more accurately,
Alexander-Kolmogoroff) cohomology, it also holds for Čech cohomology
since the two are naturally equivalent [2]. The theorem also holds for
topological pairs, as a straightforward modification of our proof will show.
We restrict our attention to the single space case for simplicity of
presentation.

In case T is an arc, the theorem is well known (no restriction on G),
following either from its truth in the Čech theory [1] and the equivalence
of the Alexander and Čech theories [2] or from a direct proof for the
Alexander theory ([6] or [5, p. 314]). On the other hand, it is known that
if X is compact, the theorem holds with T any connected space (again, no
restriction on G) as an easy generalization of the proof in [4, p. 416] shows.
The freedom of choosing the parameter space to be continuum, which is
allowed by taking X to be compact, has proved very useful in the study

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of topological semigroups. Our theorem retains this freedom of choice of parameter space while removing the compactness restriction on $X$, provided one chooses an appropriate coefficient group. It is a trivial observation, of course, that one does not really need to assume $T$ to be compact, only that each pair of points of $T$ is contained in some compact, connected subset of $T$ (e.g., $T$ locally compact, locally connected, connected). For further discussion of the problem the reader is referred to the review by K. Hofmann in Mathematical Reviews 32 (1966), p. 514.

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2. Preliminaries. In this section we collect several preliminary results we will need later.

If $X$ is a set and $\mathcal{U}$ is a collection of sets, we denote by $X_{\mathcal{U}}$ the simplicial complex whose simplexes are the finite nonempty subsets of $X$ which are contained in some member of $\mathcal{U}$.

**Proposition 1.** For each space $X$ and open cover $\mathcal{U}$ of $X$ there is a homomorphism $i_{\mathcal{U}}: H^*(X_{\mathcal{U}}) \to H^*(X)$ such that $\{H^*(X), i_{\mathcal{U}}\}$ is a direct limit of the direct system $\{H^*(X_{\mathcal{U}}), i_{\mathcal{U}}, \text{cov } X\}$ where $i_{\mathcal{U}, \mathcal{U}'}: X_{\mathcal{U}} \subseteq X_{\mathcal{U}'}$ (i.e., $\mathcal{U}$ refines $\mathcal{U}'$). Furthermore, if $f: X \to Y$ is a map and if $\mathcal{U}$ and $\mathcal{V}$ are covers of $X$ and $Y$, respectively, such that $\{f(U) \mid U \in \mathcal{U}\}$ refines $\mathcal{V}$, then the diagram

$$
\begin{array}{ccc}
H^*(Y) & \xrightarrow{f^*} & H^*(X) \\
| & i_{\mathcal{V}} & |
\end{array}
$$

commutes, where $f_0: X_{\mathcal{U}} \to Y_{\mathcal{V}}$ is the simplicial map induced by $f$.

For a proof of this proposition, see [5, p. 312].

Although the functor $H^*$ does not, in general, commute with inverse limits, the following proposition holds. Its proof is an appropriate modification of that given for the special case of exact sequences in [3, p. 226].

**Proposition 2.** On either the category of cochain complexes over compact abelian topological groups or the category of cochain complexes over finite-dimensional vector spaces over a fixed field, the functor $H^*$ commutes with inverse limits.

For a simplicial complex $K$, $C^p(K; G)$ is defined as $\{\varphi \mid \varphi: K_p \to G\}$, where $K_p = \{(x_0, \cdots, x_p) \mid \{x_0, \cdots, x_p\} \in K\}$. $C^p(K; G)$ is then just a
cartesian product of copies of $G$. Hence, we have the following propositions. The proof of the continuity, resp. linearity, of $\delta$ and $f^#$ is straightforward.

**Proposition 3.** If $G$ is a compact abelian topological group and $K$ is a simplicial complex, then $C^p(K; G)$ is a compact abelian topological group. Furthermore, $\delta : C^p(K; G) \to C^{p+1}(K; G)$ and, for a simplicial map $f : K \to K', f^# : C^p(K'; G) \to C^p(K; G)$ are continuous.

**Proposition 3'.** If $G$ is a finite-dimensional vector space and $K$ is a finite simplicial complex, then $C^p(K; G)$ is a finite-dimensional vector space. Furthermore, $\delta : C^p(K; G) \to C^{p+1}(K; G)$ and, for a simplicial map $f : K \to K', f^# : C^p(K'; G) \to C^p(K; G)$ are linear transformations.

The proof of the following proposition is simple and hence omitted.

**Proposition 4.** Suppose $K$ is a simplicial complex and $\mathcal{C}$ is a collection of subcomplexes of $K$ satisfying

(i) $\bigcup \mathcal{C} = K,$ and

(ii) if $A, B \in \mathcal{C},$ then $A \cup B$ is contained in some member of $\mathcal{C}.

Then, for any abelian group $G$, \{C*(K; G), p_A\} is an inverse limit of the inverse system \{C*(A; G), i_A.B, \mathcal{C}\}, where $i_{A,B} : A \subset B$ and $p_A : A \subset K$.

3. **Proof of the Theorem.** We prove the following Lemma, which is easily seen to be equivalent to the theorem.

**Lemma.** Suppose that $X$ and $T$ are spaces with $T$ compact and connected and that $G$ is either a compact abelian topological group or a finite-dimensional vector space. $x T$ is defined for each $t$ in $T$ by $X_t(x) = (x, t)$, then $X_t = X^s : H^*(X \times T; G) \to H^*(X; G)$ for all $r, s$ in $T$.

**Proof.** Let $g \in H^p(X \times T)$. We wish to show that $X_t^s(g) = X_t^s(g)$. From Proposition 1 we have that there is an open cover $\mathcal{U}$ of $X \times T$ and $g_{q} \in H^p((H \times T)_{q})$ such that $i_{q}(g_{q}) = g$, where $i_{q}$ is the canonical injection. Using the compactness of $T$, we get for each $x \in X$ an open set $U_x$ about $x$ and an open cover $D_{U_x}$ of $T$ such that \{$U_x \times D \mid D \in D_{U_x}$\} refines $\mathcal{U}$. We let $\mathcal{U}_x$ denote the open cover \{$U_x \mid x \in X$\} of $X$ and let $\mathcal{K}$ denote the collection of all ordered pairs $(\mathcal{C}, F)$ where $\mathcal{C}$ is a finite subcollection of $\mathcal{U}$ and $F$ is a finite subset of $X$ with $F \subset \bigcup \mathcal{C}$. By defining $(\mathcal{C}_1, F_1) \leq (\mathcal{C}_2, F_2)$ iff $\mathcal{C}_1 \subset \mathcal{C}_2$ and $F_1 \subset F_2$, $\mathcal{K}$ becomes a directed set. It is clear that, for each $t$ in $T$, $\lambda_t$ induces simplicial maps $\lambda_t^{q} : X_{q} \to (X \times T)_{q}$ and $\lambda_t^{K} : F_{\mathcal{C}} \to (X \times T)_{q}$, $K = (\mathcal{C}, F) \in \mathcal{K}$.

We show first that, for each $K = (\mathcal{C}, F) \in \mathcal{K}$, the chain maps $(\lambda_t^{K})^#$, $\lambda_t^{K} : C^*((X \times T)_{q}) \to C^*(F_{\mathcal{C}})$ are chain homotopic. Let $K = (\mathcal{C}, F) \in \mathcal{K}$
with \( \mathcal{C} = \{ U_1, \ldots , U_n \} \) and let \( \mathcal{D} \) be a common open refinement of \( \mathcal{D} U_1, \ldots , \mathcal{D} U_n \) covering \( T \). Since \( T \) is connected there exist \( D_1, \ldots , D_k \in \mathcal{D} \) and \( p_0, \ldots , p_k \) such that \( r = p_0 \in D_1, s = p_k \in D_k \) and \( p_i \in D_i \cap D_{i+1} \), \( 1 \leq i \leq k - 1 \). For each \( i = 1, \ldots , k \), \( \lambda_{p_i}^K \) and \( \lambda_{p_i-1}^K : F_\mathcal{C} \rightarrow (X \times T)_\mathcal{V} \) are contiguous since, for each \( i = 1, \ldots , n \), \( \lambda_{p_i}^K(U_i) = U_i \times \{ p_i \} \subset U_i \times D_i \) and \( \lambda_{p_i-1}^K(U_i) = U_i \times \{ p_{i-1} \} \subset U_i \times D_i \) with \( U_i \times D_i \subset U_i \times D \subset V \) for some \( D \in \mathcal{D} U_i \) and \( V \in \mathcal{U} \). It follows that, for each \( i = 1, \ldots , k \), \( (\lambda_{p_i}^K)^# \) and \( (\lambda_{p_i-1}^K)^# \) are chain homotopic and hence so are \( (\lambda_r^K)^# \) and \( (\lambda_s^K)^# \).

We now complete the proof of the Lemma. It is immediate from Proposition 4 that \( \{ C^*(X_{\mathcal{C}_1}), i_{K'}^* \} \) is an inverse limit of the inverse system \( \{ C^*(F_{\mathcal{C}_1}), i_{K,K'}^{K} \} \) where \( i_{K':F_{\mathcal{C}_1} \subset X_{\mathcal{C}_1} \text{ and } i_{K,K}:F_{\mathcal{C}_1} \subset F'_{\mathcal{C}_1} \}, K \leq K', K = (\mathcal{C}, F), K' = (\mathcal{C}', F') \). From Propositions 2, 3, and 3' we then have that \( \{ H^p(X_{\mathcal{C}_1}), i_{K}^* \} \) is an inverse limit of the system \( \{ H^p(F_{\mathcal{C}_1}), i_{K,K'}^{K} \} \). For each \( t \in T \) and \( K, K' \in \mathcal{K}, K \leq K' \), the diagrams

\[
\begin{array}{ccc}
H^p((X \times T)_{\mathcal{V}1}) & \xrightarrow{\lambda_{t}^K} & H^p((X \times T)_{\mathcal{V}1}) \\
\downarrow i_{K,K'}^* & & \downarrow i_{K,K'}^* \\
H^p(F_{\mathcal{C}_1}) & \xleftarrow{(\lambda_t^K)^*} & H^p(F_{\mathcal{C}_1})
\end{array}
\]

and

\[
\begin{array}{ccc}
H^p((X \times T)_{\mathcal{V}1}) & \xrightarrow{\lambda_{t}^{1\mathcal{U}}} & H^p((X \times T)_{\mathcal{V}1}) \\
\downarrow i_{K}^* & & \downarrow i_{K}^* \\
H^p(F_{\mathcal{C}_1}) & \xleftarrow{(\lambda_t^{1\mathcal{U}})^*} & H^p(F_{\mathcal{C}_1})
\end{array}
\]

commute. But \( (\lambda_r^K)^* = (\lambda_s^K)^* \) for each \( K \in \mathcal{K} \) since \( (\lambda_r^K)^# \) and \( (\lambda_s^K)^# \) are chain homotopic, so from the uniqueness condition in the characteristic property of inverse limits one has \( (\lambda_r^{1\mathcal{U}})^* = (\lambda_s^{1\mathcal{U}})^* \). From this and the commutativity of

\[
\begin{array}{ccc}
H^p(X \times T) & \xrightarrow{\lambda_{t}^*} & H^p(X) \\
\downarrow i_{\mathcal{V}1} & & \downarrow i_{\mathcal{V}1} \\
H^p((X \times T)_{\mathcal{V}1}) & \xleftarrow{(\lambda_t^{1\mathcal{U}})^*} & H^p((X \times T)_{\mathcal{V}1})
\end{array}
\]

(Proposition 1) one obtains \( \lambda_r^*(g) = \lambda_s^*(g) \), completing the proof.
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