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STABLE SPLITTING OF $K(G, 1)$

RICHARD HOLZSAGER

ABSTRACT. The splitting of the cohomology (ordinary and generalized) of $K(G, 1)$, for finite abelian G, is realized topologically by taking suspensions.

The cohomology of $K(Z_p, 1)$, both ordinary and K-theoretic, splits over all operations into a direct sum of $p-1$ components. This suggests asking whether the space itself splits, at least stably. The answer, in a somewhat more general situation, turns out to be affirmative. The trick used to obtain this splitting is probably at least as interesting as the result.

THEOREM. *There are simply connected spaces* $X_1, \cdots, X_{p-1}$ *and a homotopy equivalence from the suspension* $SK(Z_{p^n}, 1)$ *to the one-point union* $X_1 \vee \cdots \vee X_{p-1}$ *such that* X_i *has homology only in dimensions of the form* $2k(p-1)+2i$.

PROOF. Let r be an integer representing a generator of the multiplicative group of units in Z_p. Multiplication by r defines an isomorphism of Z_{p^n}, and hence a homotopy equivalence $f: K \to K$ ($K = K(Z_{p^n}, 1)$). This map induces multiplication by r^i on $H^{2i}(K; Z) = Z_{p^n}$. Now, for any integer s, let $g_s: SK \to SK$ be s times (with respect to the suspension structure) the identity map. The induced map on cohomology is multiplication by s in every dimension. Then for $s = r^j$, $h_j = Sf - g_s$ induces multiplication by $r^i - r^j$ on $H^{2i+1}(SK)$, thus also on $H_{2i}(SK)$. This multiplication is an isomorphism when i and j represent different classes in Z_{p-1} and has nontrivial kernel when $i \equiv j \pmod{p-1}$. For $1 \leqq i \leqq p-1$, let $m_i = h_1 \circ \cdots \circ \hat{h}_i \circ \cdots \circ h_{p-1}$. This composition induces isomorphisms on homology in dimensions of the form $2k(p-1)-2i$, but not on any other nontrivial positive dimensional homology. Now, using mapping cylinders, we can form the direct limit X_i of the sequence $SK \to SK \to SK \to \cdots$ (all maps being m_i) in such a way that the limit space is a CW-complex with homology isomorphic to the direct limit of the homology of the individual spaces, i.e., $H_j(X_i) = 0$ unless j has the form

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$2k(p-1)+2i$, while in these dimensions the map $SK \to X_1$ (from one of the terms in the direct system to the limit) induces isomorphisms. Using the suspension structure on SK, we can add the maps for $i = 1, \cdots, p-1$ to get the required $SK \to X_1 \vee \cdots \vee X_{p-1}$.

COROLLARY. *For any finite abelian group G, $S^m K(G, 1)$ has the homotopy type of a one-point union of $\sum S_p$ nontrivial spaces, where m is the number of summands in the cyclic decomposition of G and S_p is the number of elements in G with order exactly p.*

PROOF. If $G = G_1 \times \cdots \times G_n$ is the primary decomposition of G, then the map $K(G_1, 1) \vee \cdots \vee K(G_n, 1) \to K(G_1, 1) \times \cdots \times K(G_n, 1) = K(G, 1)$ is a homology equivalence by the Künneth theorem, so its suspension is a homotopy equivalence. This reduces to the case of a p-group $G = Z_{p^{i_1}} \times \cdots \times Z_{p^{i_r}}$. But since $K(G, 1) = \prod_{j=1}^{r} K(Z_{p^{i_j}}, 1)$, the result follows from the general homotopy equivalence $S(K_1 \times \cdots \times K_r) \simeq \bigvee S(K_{j_1} \wedge \cdots \wedge K_{j_s})$ (union taken over all sequences $1 \leqq j_1 < \cdots < j_s \leqq r$), together with the theorem. (Note: m is the maximum value of r for the primes occurring in G, while $S_p = p^r - 1$.)

REMARK. For $K(Z_p, 1)$, the stable splitting was conjectured by P. S. Green.

DEPARTMENT OF MATHEMATICS AND STATISTICS, AMERICAN UNIVERSITY, WASHINGTON, D.C. 20016