THE COUNTING VECTOR OF A SIMPLE GAME

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Abstract. The counting vector of a simple game is the vector 
f = (f(1), f(2), · · · , f(n)) where f(i) is the number of winning coalitions containing the player i. In this paper, we show that the counting vector of a weighted majority game determines the game uniquely. With the aid of the counting vector we find an upper bound on the number of weighted majority games.

1. Preliminaries on simple games. A simple game is a pair G = (N; W), where N = {1, 2, · · · , n} is a set of n members and W is a set of subsets of N. The members of N are called players; subsets of N are called coalitions. The elements of W are called winning coalitions. A simple game is called monotone if every superset of a winning coalition is itself a winning coalition. A weighted majority game is a simple game for which there exist n nonnegative numbers w1, w2, · · · , wn and a positive number q, such that, S is a winning coalition if and only if w(S) = ∑i∈S wi ≥ q. w = [w1, w2, · · · , wn; q] is called the representation of the game. A weighted majority game is denoted by G = (N; w) where w is its representation.

G is called constant-sum if for each coalition S exactly one of the two coalitions S and N − S is winning.

2. The counting vector theorem. Given a simple game G = (N; W), we denote by f(i) the number of winning coalitions containing player i, and by f the vector f = (f(1), f(2), · · · , f(n)) called the counting vector of the game.

Theorem 2.1. Let

G1 = (N; [w1(1), w2(1), · · · , wn(1); q(1)])

and

G2 = (N; [w1(2), w2(2), · · · , wn(2); q(2)])

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be two weighted majority games. Their counting vectors $f_1$ and $f_2$ are equal if and only if $G_1 = G_2$.

**Proof.** Suppose that $G_1 \neq G_2$ but $f_1 = f_2$, i.e., for every $i \in N$, 
\[(2.1) \quad f_i(i) = f_i'(i). \]

(We may assume that $f \neq 0$ since otherwise we have the trivial game $(N; \phi)$.) Let $W_1, W_2$ be the sets of winning coalitions in $G_1, G_2$ respectively. We set $\Omega_1 = W_1 - W_2$, $\Omega_2 = W_2 - W_1$. If $W_k \subseteq W_p$ ($k, p = 1, 2$, $k \neq p$), then $f_k(i) \leq f_p(i)$ for every $i$ and inequality holds for at least one player $i_0$, in contradiction to (2.1); hence,
\[(2.2) \quad \Omega_1 \neq \phi, \quad \Omega_2 \neq \phi. \]

For each $S \in \Omega_1$ and $T \in \Omega_2$ we have
\[(2.3) \quad w^{(1)}(S) > w^{(1)}(T). \]

Summing these inequalities over $\Omega_2$ for a fixed $S \in \Omega_1$, we have
\[(2.4) \quad |\Omega_2| \sum w^{(1)}(S) > \sum_{T \in \Omega_2} w^{(1)}(T) \]

where $|\Omega|$ is the number of elements of $\Omega$. Summing all the inequalities (2.4) over $\Omega_1$, we have
\[(2.5) \quad |\Omega_2|| \sum_{S \in \Omega_1} w^{(1)}(S) > |\Omega_1|| \sum_{T \in \Omega_2} w^{(1)}(T). \]

Let $\phi_k(i)$ be the number of coalitions in $\Omega_k$ containing player $i$. Since $\Omega_1 = W_1 - (W_1 \cap W_2)$ and $\Omega_2 = W_2 - (W_1 \cap W_2), f_1(i) = f_2(i)$ implies $\phi_1(i) = \phi_2(i)$. Denoting the common value by $\phi(i)$, we have
\[(2.6) \quad \sum_{S \in \Omega_1} w^{(1)}(S) = \sum_{i \in N} \phi(i)w_i^{(1)}, \]
\[(2.7) \quad \sum_{T \in \Omega_2} w^{(1)}(T) = \sum_{i \in N} \phi(i)w_i^{(1)}. \]

Since $q^{(1)}$ is positive and $\Omega_1 \neq \phi, \Omega_2 \neq \phi$, we have, from (2.5), (2.6), and (2.7),
\[(2.8) \quad |\Omega_1| < |\Omega_2|. \]

Using the representation $[w_1^{(2)}, w_2^{(2)}, \ldots, w_n^{(2)}, q^{(2)}]$ of $G_2$ we have
\[(2.9) \quad |\Omega_2| < |\Omega_1| \]

in contradiction to (2.8) and hence $f_1 \neq f_2$. 
Corollary 2.1. Let \( G_1 = (N; W_1) \) be a weighted majority game and \( G_2 = (N; W_2) \) a simple game. If \( G_1 \not\cong G_2 \) and

\[
(2.10) \quad |W_1| \geq |W_2|,
\]

then \( f_1 \not\cong f_2 \).

Proof. \( (2.10) \) implies

\[
(2.11) \quad |\Omega_1| \geq |\Omega_2|.
\]

If \( f_1 = f_2 \) then as in the proof of Theorem 2.1, \( |\Omega_1| < |\Omega_2| \). Hence \( f_1 \not\cong f_2 \).

Corollary 2.2. Let \( G_1 \) be a constant-sum weighted majority game and \( G_2 \) a constant-sum simple game. If \( G_1 \not\cong G_2 \), then \( f_1 \not\cong f_2 \).

Proof. In any constant-sum simple game, the number of winning coalitions is \( 2^{n-1} \) (\( n \) being the number of players), hence \( |W_1| = |W_2| \). The assumptions of Corollary 2.1 are valid, hence \( f_1 \not\cong f_2 \).

3. An upper bound on the number of weighted majority games. In [1], J. R. Isbell raised the problem of finding an upper bound on the number of weighted majority games. No answer was given, except for the bound \( 2^n \), which is the number of sets of coalitions in \( N \). Using the counting vector concept a smaller value is obtainable for the upper bound.

It is evident that for every \( i, 0 \leq f(i) \leq 2^{n-1} \). If \( f(i) = 0 \) for some \( i \), then \( N \notin W \); hence \( W = \emptyset \), i.e., the game is the trivial game \( G_0 = (N; \emptyset) \). For all other games \( 1 \leq f(i) \leq 2^{n-1} \). The number of vectors

\[
(3.1) \quad a = (a_1, a_2, \cdots, a_n), \quad a_i = 1, 2, \cdots, 2^{n-1}, \quad i = 1, 2, \cdots, n,
\]

is \( 2^{n(n-1)} \). Since not every vector of the type \( (3.1) \) is a counting vector, there are less than \( 2^{n(n-1)} \) weighted majority \( n \)-person games.

Two games, \( G_1 = (N; W_1) \) and \( G_2 = (N; W_2) \), are said to be equivalent if there exists a permutation \( \pi \) of \( N \) such that

\[
(3.2) \quad W_2 = \{\pi S: S \in W_1\}
\]

where \( \pi S = \{\pi(i): i \in S\} \).

It is evident that this relation is an equivalence. If we identify equivalent games, \( f \) is no longer a vector but an unordered set of \( n \) numbers. Hence their number is less than \( (2^{n-1}+n-1) \), the number of \( n \)-selections of \( 2^{n-1} \) given elements.

The following example shows that there exist two nonequivalent monotone games which have the same counting vector.

Example. Let \( N \) be a set of seven players \( \{1, 2, 3, 4, 5, 6, 7\} \). \( W_1 \) consists of all 4, 5, 6, and 7 player coalitions and of the following 3 player coalitions: \( (2, 5, 3), (3, 7, 1), (1, 4, 2), (2, 6, 7), (3, 6, 4), (1, 6, 5) \), and
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(4, 5, 7). $W_2$ consists of all 4, 5, 6, and 7 player coalitions and of the following 3 player ones: (2, 4, 3), (3, 7, 1), (1, 5, 2), (2, 6, 7), (3, 6, 4), (1, 6, 5), and (4, 5, 7).

The two games $(N; W_1)$ and $(N; W_2)$ are monotone and it is easy to see that they have the same counting vector. However, $W_1$ contains no two disjoint 3 player coalitions while $W_2$ does, and thus they are not equivalent.

4. The desirability relation. Two players $i$ and $j$, in a simple game $(N; W)$ are called symmetric if for every coalition $S$ not containing $i$ nor $j$, $S \cup \{i\}$ is winning if and only if $S \cup \{j\}$ is winning; this is denoted by $i \sim j$. If for every coalition $S$ not containing $i$ nor $j$, $S \cup \{i\} \in W$ implies $S \cup \{j\} \in W$, then $i$ is said to be more desirable than $j$; this is denoted by $i \succeq j$.

If $i \succeq j$ and $j \succeq i$ then $i \sim j$. If $i \succeq j$ but $i$ and $j$ are not symmetric, we say that $i$ is strictly more desirable than $j$; this is denoted by $i > j$. In a weighted majority game with a representation $[w_1, w_2, \ldots, w_n; q]$, $w_1 \succeq w_j$ implies $i \succeq j$ and there exists a representation $[w_1^*, w_2^*, \ldots, w_n^*; q^*]$ such that $i \sim j$ implies $w_i^* = w_j^* [3]$. The counting vector strictly preserves the desirability relation, i.e., $f(i) > f(j)$ implies $i > j$ and $f(i) = f(j)$ implies $i \sim j [2]$. This enables us to reduce the number of unknowns when seeking a representation of a given weighted majority game.

REFERENCES


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