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ALGEBRAIC SOLUTION OF $x^y = y^x$ $(0 < x < y)$

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ABSTRACT. The characterization of pairs of algebraic numbers (algebraic integers) which are commutative with respect to the exponentiation will be given.

It is well known that there is only one integer solution of $x^y = y^x$ $(0 < x < y)$, namely $2^4 = 4^2$. We give a simple proof of

THEOREM. *The solutions of* $x^y = y^x$ $(0 < x < y)$ *in the real algebraic field are parametrized by* $x = s^{s/(s-1)}$, $y = s^{1/(s-1)}$ *where* s *is any rational number in the open unit interval* $0 < s < 1$ *and the solutions in the ring of algebraic integers are parametrized by* $x = (k+1)^{1/k}$, $y = (k+1) \times (k+1)^{1/k}$ *where* $k = 1, 2, 3, \cdots$.

PROOF. It is easy to see that the numbers x and y given here are algebraic numbers (integers) satisfying the condition $x^y = y^x$ $(0 < x < y)$. Conversely, let $0 < x < y$ be an algebraic solution then $s = x/y$, $u = s/(s-1) = x/(x-y)$ and $v = 1/(s-1) = y/(x-y)$ are all algebraic numbers. If s were an irrational algebraic number then so would be u and v. This is impossible because, since $s^u = x$ and $s^v = y$, x and y would be transcendental by the Gelfond-Schneider theorem. Therefore $0 < s < 1$ is a rational number. Let $s = m/n$ where $0 < m < n$ are rational integers with $(m, n) = 1$. If x is an algebraic integer then so is

$$x^{n-m} = s^{u(n-m)} = s^{s(n-m)/(s-1)} = n^m/m^m \quad \text{with } (n^m, m^m) = 1.$$

Therefore $m = 1$ (so $n \geqq 2$) and we get $x = n^{1/(n-1)}$, $y = n^{n/(n-1)}$ where $n = 2, 3, \cdots$.

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