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MODULES OVER THE ENDOMORPHISM RING OF A FINITELY GENERATED PROJECTIVE MODULE

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ABSTRACT. Let P_R be a projective module with trace ideal T. An R-module X_R is T-accessible if $XT = X$. If P_R is finitely generated projective and C is the R-endomorphism ring of P_R, such that ${}_CP_R$, then for X_R, Hom $(P_R, X_R)_C$ is artinian (noetherian) if and only if X_R satisfies the minimum (maximum) condition on T-accessible submodules. Further, if X_R is T-accessible then Hom $(P_R, X_R)_C$ is finitely generated if and only if X_R is finitely generated.

The purpose of the present paper is to investigate ${}_C\text{Hom}\,({}_CP_R, X_R)$, where P_R is a finitely generated projective R-module and $C = \text{End}\,(P_R)$, the R-endomorphism ring of P, with respect to the properties of chain conditions and finite generation. Throughout this paper R is a ring with identity and all modules over R are unitary. The convention of writing module-homomorphisms on the side opposite the scalars is adopted here.

1. **Preliminaries.** Let P_R be a finitely generated projective R-module with $C = \text{End}\,(P_R)$ such that ${}_CP_R$. The dual module of P_R is (with respect to R_R), ${}_RP^*_C = \text{Hom}\,({}_CP_R, {}_RR_R)$. It is well known, see [1], that the map $P_R \xrightarrow{\delta_P} P^{**}_R = \text{Hom}\,({}_RP^*, {}_RR)$ given by $p \to \hat{p}$, where $f\hat{p} = fp$, for $f \in P^*$ is an R-isomorphism.

LEMMA 1.1. *For P_R finitely generated projective, $C = \text{End}\,(P_R)$, the map* $\text{End}\,(P_R) \to \text{End}\,({}_RP^*)$ *given by $c \to \bar{c}$, where $f\bar{c} = fc$ is a ring isomorphism.*

PROOF. The above map is nothing more than the composite of the following maps

$$\begin{array}{ccc} \text{Hom}\,(P_R, P_R) & \xrightarrow{\text{Hom}\,(1,\, \delta_P)} & \text{Hom}\,(P_R, P^{**}_R) \\ & & \Big\downarrow t \\ & & \text{Hom}\,({}_RP^*, {}_RP^*) \end{array}$$

where t is the natural equivalence of functors in [2, Chapter II, Exercise 4].

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Since δ_P is an isomorphism the lemma follows. Let P_R be a projective R-module and T the trace ideal of P. It is well known, e.g., see [3], that T is an idempotent two-sided ideal of R and $PT = P$.

Some results on the trace ideal of a projective module are listed in the proposition below.

Proposition 1.2. *Let P_R be a projective module with trace ideal T, then*:

(i) *For X_R*, Hom $(P_R, X_R) = 0$ *if and only if* $XT = 0$.

(ii) *For X_R, $XT = X$ if and only if X_R is an epimorphic image of a direct sum (coproduct) of copies of P_R.*

The proof of this proposition is an easy consequence of the definition and is left to the reader.

2. **Main results.** Throughout this section P_R denotes a finitely generated projective module, T its trace ideal, $C = \text{End}\,(P_R)$, ${}_RP^*_C = \text{Hom}\,({}_CP_R, {}_RR_R)$ and, for X_R, $X'_C = \text{Hom}\,({}_CP_R, X_R)$.

Definition 2.1. (a) A module X_R is T-accessible if and only if $XT = X$.

(b) A module X_R is T-artinian (T-noetherian) if and only if X_R satisfies the minimum (maximum) condition on T-accessible submodules.

For a module X_R and M_C a submodule of X'_C, MP denotes the submodule of X_R generated by the images of P_R by homomorphisms in M_C. If Y_R is a submodule of X_R then Hom $({}_CP_R, Y_R) = Y'_C$ can be identified with a submodule of X'_C since Hom $(P_R, —)$ is (left) exact.

It is now clear that a submodule Y_R of X_R is T-accessible if and only if $Y = MP$ for some submodule M_C of X'_C. These facts will be used in what follows.

Theorem 2.2. *For a module X_R, the correspondence $M_C \leftrightarrow MP$ is a one-to-one correspondence, inclusion preserving, between the submodules of X'_C and the T-accessible submodules of X_R.*

Proof. Since MP is an R-submodule of X_R, the theorem will follow if Hom $(P_R, MP_R) = M$. Clearly $M \subseteq \text{Hom}\,(P_R, MP_R)$. Let $f \in \text{Hom}\,(P_R, MP_R)$, then let ${}^MP$ be a direct sum of M copies of P_R and denote by $\pi_m : {}^MP \to P$ the mth projection map. Now the following diagram of R-modules and R-homomorphisms is commutative

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle f'}\swarrow & \downarrow{\scriptstyle f} & & \\ {}^MP & \xrightarrow{\mu} & MP & \longrightarrow & 0 \quad \text{(exact)} \end{array}$$

where $\mu x = \sum_{m \in M} m(\pi_m x)$ and f' exists since P_R is projective.

Since P is finitely generated there is a finite subset N of M such that $\pi_m f'p = 0$ for all $p \in P$ and all $m \notin N$. Now

$$fp = \mu f'p = \sum_{m\in N} m(\pi_m f'p) = \left(\sum_{m\in N} m(\pi_m f')\right) p$$

hence $f = \sum_{m\in N} m(\pi_m f')$. Since $\pi_m f' \in C$, $f \in M_C$ and the theorem follows.

Corollary 1. *A module X_R is T-artinian (T-noetherian) if and only if X'_C is artinian (noetherian).*

Corollary 2. *If X_R is T-accessible then X_R is finitely generated if and only if X'_C is finitely generated.*

Proof. A well-known characterization of finitely generated modules is the following: A module X_R is finitely generated if and only if every totally ordered subset, by inclusion, of proper submodules of X_R has a proper submodule of X_R for its union (least upper bound). By the theorem since X_R is T-accessible X'_C is finitely generated if and only if the union of a totally ordered set of proper T-accessible submodules of X_R is a proper submodule of X_R, hence if X_R is finitely generated so is X'_C.

Conversely, suppose X'_C is finitely generated and $\{Y_i\}_I$, I some index set, is a totally ordered subset of proper submodules of X_R. If $\bigcup_I Y_i = X$, then $\bigcup_I TY_i = TX = X$, hence it is sufficient to show that $\bigcup_I TY_i \neq X$. If $\bigcup_I TY_i = X$ then let $(TY_i)' = M_i \subsetneqq X'_C$. Since P_R is finitely generated it follows that $X'_C = \text{Hom}(P_R, \bigcup TY_i) = \bigcup_I M_i$, a contradiction to the finite generation of X'_C since $M_i \neq X'$, for if $M_i = X'$, $M_i P = TY_i = X$ and the corollary follows.

Corollary 3. *If $0 \to U_R \to V_R \to W_R \to 0$ is exact then V_R is T-artinian, respectively T-noetherian, if and only if U_R and W_R are T-artinian, respectively T-noetherian.*

Proof. Since P_R is projective $0 \to U'_C \to V'_C \to W'_C \to 0$ is exact and the corollary follows from an analogous result for modules and Corollary 1.

It is well known, e.g., [4], that the functors Hom $({}_CP_R, X_R)$ and $X \otimes_R P^*_C$ are naturally equivalent as functors of X_R, hence all previous results can be stated replacing X'_C with $X \otimes_R P^*_C$.

In view of Lemma 1.1, ${}_RP^*_C = \text{Hom}({}_CP_R, {}_RR_R)$, an endomorphism of ${}_RP^*$ is given by a unique $c \in C$, namely if $d \in \text{End}({}_RP^*)$, there is a unique $c \in C$ such that $fd = fc$ for every $f \in P^*$. With this identification the following is valid.

Corollary 4. *${}_RT$ is finitely generated if and only if ${}_CP$ is finitely generated.*

PROOF. ${}_CP \cong \operatorname{Hom}({}_RP^*_C, {}_RT)$ so by Corollary 2, the above follows.

Some obvious results of the preceding are listed in the next proposition without proof.

PROPOSITION 2.3. *If P_R, C are as in Theorem 2.2, then*

(i) *If R_R is artinian (noetherian), so is C_C.*

(ii) *If R_R is artinian (noetherian), so is P^*_C.*

(iii) *If X_R is artinian (noetherian), so is* $\operatorname{Hom}({}_CP_R, X_R)_C$.

Now will be taken up the problem of whether direct sums in the correspondence of Theorem 2.2 are preserved. Since P_R is finitely generated, it follows that if $\sum_I X_i$ is a direct sum of submodules of X_R, then $(\sum_I X_i)' = \sum_I X_i'$ is a direct sum of C submodules of X'_C.

For the question of whether $\sum_I M_iP = (\sum_I M_i)P$ is a direct sum whenever $\sum_I M_i$ is a direct sum of C submodules of X'_C, the following notion will be useful.

DEFINITION 2.4. For the module X_R, T is X-faithful if $xT \neq 0$ for each $0 \neq x \in X$.

THEOREM 2.5. *Let X_R be such that T is X-faithful, then if $\sum_I M_i$ is a direct sum of C submodules of X'_C, then $\sum_I M_iP$ is a direct sum of R submodules of X_R.*

PROOF. For an index $j \in I$,

$$\left[M_jP \cap \left(\sum_{i \neq j} M_iP\right)\right]' \subseteq (M_jP)' \cap \left(\sum_{i\neq j} M_iP\right)' \subseteq M_j \cap \sum_{i \neq j} M_i = 0,$$

where the last inclusion follows from the facts that $(M_iP)' = M_i$ and since P_R is finitely generated,

$$\left(\sum_{i\neq j} M_iP\right)' \subseteq \sum_{i \neq j}(M_iP)' = \sum_{i\neq j} M_i.$$

Now by Proposition 1.1,

$$T\left[M_jP \cap \left(\sum_{i\neq j}(M_iP)\right)\right] = 0$$

and since T is X-faithful, $M_iP \cap (\sum_{i\neq j} M_iP) = 0$ and the theorem follows.

COROLLARY 1. *If T is X_R-faithful and X_R has finite Goldie dimension, see* [3], *X'_C has finite Goldie dimension.*

COROLLARY 2. *If ${}_RT$ is faithful with finite Goldie dimension, then C_C has finite Goldie dimension.*

PROOF. It is sufficient to show T is P-faithful. Since P_R is a direct summand of a free R-module, if $xT = 0$ for some $0 \neq x \in P$, $rT = 0$ for some $0 \neq r \in R$, a contradiction, so T is P-faithful.

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