MODULES OVER THE ENDMORPHISM RING OF A FINITELY GENERATED PROJECTIVE MODULE

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Abstract. Let $P_R$ be a projective module with trace ideal $T$. An $R$-module $X_R$ is $T$-accessible if $XT = X$. If $P_R$ is finitely generated projective and $C$ is the $R$-endomorphism ring of $P_R$, such that $cP_R$, then for $X_R$, $\text{Hom}(P_R, X_R)c$ is artinian (noetherian) if and only if $X_R$ satisfies the minimum (maximum) condition on $T$-accessible submodules. Further, if $X_R$ is $T$-accessible then $\text{Hom}(P_R, X_R)c$ is finitely generated if and only if $X_R$ is finitely generated.

The purpose of the present paper is to investigate $\mathcal{O}\text{Hom}(C_P, X_R)$, where $P_R$ is a finitely generated projective $R$-module and $C = \text{End}(P_R)$, the $R$-endomorphism ring of $P$, with respect to the properties of chain conditions and finite generation. Throughout this paper $R$ is a ring with identity and all modules over $R$ are unitary. The convention of writing module-homomorphisms on the side opposite the scalars is adopted here.

1. Preliminaries. Let $P_R$ be a finitely generated projective $R$-module with $C = \text{End}(P_R)$ such that $cP_R$. The dual module of $P_R$ is (with respect to $RR$), $R^{\ast}P_R = \text{Hom}(P_R, rRr)$. It is well known, see [1], that the map $P_R \rightarrow R^{\ast}P_R = \text{Hom}(R^{\ast}P_R, R)$ given by $p \rightarrow \hat{p}$, where $f\hat{p} = fp$, for $f \in P^\ast$ is an $R$-isomorphism.

Lemma 1.1. For $P_R$ finitely generated projective, $C = \text{End}(P_R)$, the map $\text{End}(P_R) \rightarrow \text{End}(R^{\ast}P_R)$ given by $c \rightarrow \hat{c}$, where $f\hat{c} = fc$ is a ring isomorphism.

Proof. The above map is nothing more than the composite of the following maps

$$
\begin{array}{ccc}
\text{Hom}(P_R, P_R) & \xrightarrow{\text{Hom}(1, \delta_P)} & \text{Hom}(P_R, P_{R}^{\ast}) \\
& & t \downarrow \\
& & \text{Hom}(R^{\ast}P_R, R^{\ast}P_R)
\end{array}
$$

where $t$ is the natural equivalence of functors in [2, Chapter II, Exercise 4].
Since $\delta_p$ is an isomorphism the lemma follows. Let $P_R$ be a projective $R$-module and $T$ the trace ideal of $P$. It is well known, e.g., see [3], that $T$ is an idempotent two-sided ideal of $R$ and $PT = P$.

Some results on the trace ideal of a projective module are listed in the proposition below.

**Proposition 1.2.** Let $P_R$ be a projective module with trace ideal $T$, then:

(i) For $X_R$, $\text{Hom} (P_R, X_R) = 0$ if and only if $XT = 0$.

(ii) For $X_R$, $XT = X$ if and only if $X_R$ is an epimorphic image of a direct sum (coproduct) of copies of $P_R$.

The proof of this proposition is an easy consequence of the definition and is left to the reader.

2. **Main results.** Throughout this section $P_R$ denotes a finitely generated projective module, $T$ its trace ideal, $C = \text{End} (P_R)$, $R^P_C = \text{Hom} (C P_R, R R)$ and, for $X_R$, $X'_C = \text{Hom} (C P_R, X_R)$.

**Definition 2.1.** (a) A module $X_R$ is $P$-accessible if and only if $XT = X$.

(b) A module $X_R$ is $P$-artinian ($P$-noetherian) if and only if $X_R$ satisfies the minimum (maximum) condition on $P$-accessible submodules.

For a module $X_R$ and $M_C$ a submodule of $X'_C$, $MP$ denotes the submodule of $X_R$ generated by the images of $P_R$ by homomorphisms in $M_C$. If $Y_R$ is a submodule of $X_R$ then $\text{Hom} (C P_R, Y_R) = Y'_C$ can be identified with a submodule of $X'_C$ since $\text{Hom} (P_R, -)$ is (left) exact.

It is now clear that a submodule $Y_R$ of $X_R$ is $P$-accessible if and only if $Y = MP$ for some submodule $M_C$ of $X'_C$. These facts will be used in what follows.

**Theorem 2.2.** For a module $X_R$, the correspondence $M_C \leftrightarrow MP$ is a one-to-one correspondence, inclusion preserving, between the submodules of $X'_C$ and the $P$-accessible submodules of $X_R$.

**Proof.** Since $MP$ is an $R$-submodule of $X_R$, the theorem will follow if $\text{Hom} (P_R, MP_R) = M$. Clearly $M \subseteq \text{Hom} (P_R, MP_R)$. Let $f \in \text{Hom} (P_R, MP_R)$, then let $MP$ be a direct sum of $M$ copies of $P_R$ and denote by $\pi_m: MP \to P$ the $m$th projection map. Now the following diagram of $R$-modules and $R$-homomorphisms is commutative

\[
\begin{array}{cccc}
P & & & \\
\downarrow f' & \downarrow f & \downarrow \mu & \\
& MP & \to & MP \to 0 \quad (\text{exact})
\end{array}
\]

where $\mu x = \sum_{m \in M} m(\pi_m x)$ and $f'$ exists since $P_R$ is projective.
Since $P$ is finitely generated there is a finite subset $N$ of $M$ such that $\pi_m f' p = 0$ for all $p \in P$ and all $m \notin N$. Now

$$fp = \mu f' p = \sum_{m \in N} m(\pi_m f' p) = \left( \sum_{m \in N} m(\pi_m f') \right) p$$

hence $f = \sum_{m \in N} m(\pi_m f')$. Since $\pi_m f' \in C$, $f \in M_C$ and the theorem follows.

**Corollary 1.** A module $X_R$ is $T$-artinian (T-noetherian) if and only if $X'_C$ is artinian (noetherian).

**Corollary 2.** If $X_R$ is $T$-accessible then $X_R$ is finitely generated if and only if $X'_C$ is finitely generated.

**Proof.** A well-known characterization of finitely generated modules is the following: A module $X_R$ is finitely generated if and only if every totally ordered subset, by inclusion, of proper submodules of $X_R$ has a proper submodule of $X_R$ for its union (least upper bound). By the theorem since $X_R$ is $T$-accessible $X'_C$ is finitely generated if and only if the union of a totally ordered set of proper $T$-accessible submodules of $X_R$ is a proper submodule of $X_R$, hence if $X_R$ is finitely generated so is $X'_C$.

Conversely, suppose $X'_C$ is finitely generated and $\{Y_i\}_I$, $I$ some index set, is a totally ordered subset of proper submodules of $X_R$. If $\bigcup_I Y_i = X$, then $\bigcup_I TY_i = TX = X$, hence it is sufficient to show that $\bigcup_I TY_i \neq X$. If $\bigcup_I TY_i = X$ then let $(TY_i)' = M_i \subseteq X'_C$. Since $P_R$ is finitely generated it follows that $X'_C = \text{Hom}(P_R, \bigcup TY_i) = \bigcup M_i$, a contradiction to the finite generation of $X'_C$ since $M_i \neq X'$, for if $M_i = X'$, $M_i P = TY_i = X$ and the corollary follows.

**Corollary 3.** If $0 \rightarrow U_R \rightarrow V_R \rightarrow W_R \rightarrow 0$ is exact then $V_R$ is $T$-artinian, respectively T-noetherian, if and only if $U_R$ and $W_R$ are $T$-artinian, respectively $T$-noetherian.

**Proof.** Since $P_R$ is projective $0 \rightarrow U'_C \rightarrow V'_C \rightarrow W'_C \rightarrow 0$ is exact and the corollary follows from an analogous result for modules and Corollary 1.

It is well known, e.g., [4], that the functors $\text{Hom}(cP_R, X_R)$ and $X \otimes_R P_C^*$ are naturally equivalent as functors of $X_R$, hence all previous results can be stated replacing $X'_C$ with $X \otimes_R P_C^*$.

In view of Lemma 1.1, $R P_C^* = \text{Hom}(cP_R, R R_R)$, an endomorphism of $R P^*$ is given by a unique $c \in C$, namely if $d \in \text{End}(R P^*)$, there is a unique $c \in C$ such that $fd = fc$ for every $f \in P^*$. With this identification the following is valid.

**Corollary 4.** $R T$ is finitely generated if and only if $cP$ is finitely generated.
\textbf{Proof.} $c_P \cong \text{Hom}_R(P, R_T)$ so by Corollary 2, the above follows. Some obvious results of the preceding are listed in the next proposition without proof.

\textbf{Proposition 2.3.} If $P_R$, $C$ are as in Theorem 2.2, then

(i) If $R_R$ is artinian (noetherian), so is $C_C$.

(ii) If $R_R$ is artinian (noetherian), so is $P_P^\ast$.

(iii) If $X_R$ is artinian (noetherian), so is $\text{Hom}_R(c_P, X_R)_C$.

Now will be taken up the problem of whether direct sums in the correspondence of Theorem 2.2 are preserved. Since $P_R$ is finitely generated, it follows that if $\sum_i X_i$ is a direct sum of submodules of $X_R$, then $(\sum_i X_i)' = \sum_i X_i'$ is a direct sum of $C$ submodules of $X_C$.

For the question of whether $\sum_i M_iP = (\sum_i M_i)P$ is a direct sum whenever $\sum_i M_i$ is a direct sum of $C$ submodules of $X_C$, the following notion will be useful.

\textbf{Definition 2.4.} For the module $X_R$, $T$ is $X$-faithful if $xT \neq 0$ for each $0 \neq x \in X$.

\textbf{Theorem 2.5.} Let $X_R$ be such that $T$ is $X$-faithful, then if $\sum_i M_i$ is a direct sum of $C$ submodules of $X_C$, then $\sum_i M_iP$ is a direct sum of $R$ submodules of $X_R$.

\textbf{Proof.} For an index $j \in I$,

\[ [M_jP \cap (\sum_{i \neq j} M_iP)]' \subseteq (M_jP)' \cap (\sum_{i \neq j} M_iP) \subseteq M_j \cap \sum M_i = 0, \]

where the last inclusion follows from the facts that $(M_jP)' = M_i$ and since $P_R$ is finitely generated,

\[ \left( \sum_{i \neq j} M_iP \right)' \subseteq \sum_{i \neq j} (M_iP)' = \sum M_i. \]

Now by Proposition 1.1,

\[ T \left[ M_jP \cap \left( \sum_{i \neq j} (M_iP) \right) \right] = 0 \]

and since $T$ is $X$-faithful, $M_jP \cap (\sum_{i \neq j} M_iP) = 0$ and the theorem follows.

\textbf{Corollary 1.} \textit{If $T$ is $X_R$-faithful and $X_R$ has finite Goldie dimension, see [3], $X_C$ has finite Goldie dimension.}

\textbf{Corollary 2.} \textit{If $R_T$ is faithful with finite Goldie dimension, then $C_C$ has finite Goldie dimension.}
PROOF. It is sufficient to show $T$ is $P$-faithful. Since $P_R$ is a direct summand of a free $R$-module, if $xT = 0$ for some $0 \neq x \in P$, $rT = 0$ for some $0 \neq r \in R$, a contradiction, so $T$ is $P$-faithful.

REFERENCES


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