A NONARCHIMEDEAN THEORY OF ANALYTIC CONTINUATION IN SEVERAL VARIABLES

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Abstract. Recently B. Dwork proved the validity of the functional equation, conjectured by A. Weil, for a nonsingular projective hypersurface defined over a finite field. The proof made use of work of M. Krasner, wherein a uniqueness theorem for an analog of analytic continuation in ultrametric spaces is proved. The methods involved give information concerning the behavior of the undetermined factor $\pm 1$ in the functional equation for such a hypersurface if one of the coefficients of the polynomial is varied. In this paper, Krasner's result is extended to a uniqueness theorem for analytic elements in $n$ variables. This result will be applied to the Weil zeta function in a later work.

1. Preliminaries. Let $\mathbb{R}$ be an algebraically closed field complete with respect to a nonarchimedean rank one valuation $x \to \text{ord } x$ with value group $G \subset R$, where $R$ denotes the additive group of real numbers. We shall assume that $G$ is dense in $R$. For $b \in R$, we define $\Gamma_b = \{ \xi \in \mathbb{R} : \text{ord } \xi = b \}$. Let $\mathfrak{O}$ denote the valuation ring of $\mathbb{R}$, $\mathfrak{O} = \bigcup_{b \in G} \Gamma_b$, and let $\mathfrak{P}$ denote the ideals of nonunits in $\mathfrak{O}$, $\mathfrak{P} = \bigcup_{b \in G} \Gamma_b$. It will occasionally be convenient to use the notation $|x| = p^{-\text{ord } x}$, where $p$ is the characteristic of the residue class field of $\mathbb{R}$, denoted by $k$.

The following definition is due to Krasner [2].

Definition 1.1. Let $D$ be a subset of the "projective field" $\mathbb{R}^* = \mathbb{R} \cup \{ \infty \}$. We say that $D$ is a quasi-connected domain of $\mathbb{R}^*$ if, for every $\alpha \in D \cap \mathbb{R}$, the following property is satisfied: for every $\xi \in D$, the set

$$H_\xi = \{ |x - \alpha| : x \in \mathbb{R} - D, |x - \alpha| < |\xi - \alpha| \}$$

is a finite set.

Lemma 1.2. Let $\zeta_1, \zeta_2, \ldots, \zeta_r$ be distinct elements of $\mathfrak{O}$; then there is an element $\xi$ of $\mathfrak{O}$ such that $|\xi - \zeta_i| = 1$ for $i = 1, 2, \ldots, r$.

This is a special case of Lemma 1 of [3], and so we may omit the proof.
Proposition 1.3. Let \( f(x) \in \mathcal{O}[x], f \neq 0 \). For any positive number \( \delta \), the sets
\[
W_\delta(f) = \{ \xi \in \mathcal{O} : |f(\xi)| > \delta \}, \quad W^\#_\delta(f) = \{ \xi \in \mathcal{O} : |f(\xi)| \geq \delta \}
\]
are quasi-connected.

Proof. Let \( f(x) = (x - \zeta_1)^{\epsilon_1} \cdots (x - \zeta_r)^{\epsilon_r}(x\beta_1 - 1)^{\epsilon_1} \cdots (x\beta_s - 1)^{\epsilon_s} \), \( \zeta_1, \zeta_2, \ldots, \zeta_r \) distinct elements of \( \mathcal{O} \), \( \beta_1, \beta_2, \ldots, \beta_s \) distinct nonunits in \( \mathcal{O} \). For \( \delta \geq 0 \), let \( R_\delta, R^\#_\delta \) be sets of real \( r \)-tuples defined by
\[
R_\delta = \{ (\delta_1, \ldots, \delta_r) : \delta_1^\epsilon_1 \cdots \delta_r^\epsilon_r > \delta, 0 \leq \delta_i \leq 1, i = 1, \ldots, r \},
\]
\[
R^\#_\delta = \{ (\delta_1, \ldots, \delta_r) : \delta_1^\epsilon_1 \cdots \delta_r^\epsilon_r \geq \delta, 0 \leq \delta_i \leq 1, i = 1, \ldots, r \},
\]
and, for any \( r \)-tuple \( (\delta_1, \ldots, \delta_r) \), let \( W(\delta_1, \ldots, \delta_r) = \{ \xi \in \mathcal{O} : |\xi - \zeta_i| \geq \delta_i, i = 1, 2, \ldots, r \} \). Since, as is clear from Definition 1.1, a disk from which finitely many (open or closed) disks have been removed is a quasi-connected domain, it follows that, for any \( r \)-tuple \( (\delta_1, \ldots, \delta_r) \), the set \( W(\delta_1, \ldots, \delta_r) \) is quasi-connected.

Let us consider the collections
\[
C_\delta = \{ W(\delta_1, \ldots, \delta_r) : (\delta_1, \ldots, \delta_r) \in R_\delta \},
\]
\[
C^\#_\delta = \{ W(\delta_1, \ldots, \delta_r) : (\delta_1, \ldots, \delta_r) \in R^\#_\delta \}.
\]
It is noted that, for any \( \delta \), \( C_\delta \) is a subfamily of \( C^\#_\delta \), and that \( C_\delta \) (respectively \( C^\#_\delta \)) is an empty family of sets if \( \delta \geq 1 \) (respectively \( \delta > 1 \)). We now recall that, in the terminology of Krasner, a family \( F \) of sets is said to be linked if any two sets \( A, B \) of \( F \) can be joined by a chain, that is to say a finite collection \( A = C_0, C_1, \ldots, C_m = B \) of sets of the family such that any two consecutive terms \( C_i, C_{i+1} \) are nondisjoint, and we assert that the collections \( C_\delta, C^\#_\delta \) are either empty or linked families of quasi-connected sets. In fact, we are able to prove a stronger statement, namely that for any choice of \( \delta \) in the closed unit interval, there is an element \( \xi \in \mathcal{O} \) common to each member of the family \( C^\#_\delta \). For, according to Lemma 1.2, an element \( \xi \) of \( \mathcal{O} \) may be chosen satisfying \( |\xi - \zeta_i| = 1, i = 1, 2, \ldots, r \), and therefore, since \( (\delta_1, \ldots, \delta_r) \in C^\#_\delta \) entails \( \delta_i \leq 1 \) for all \( i \), the assertion follows. But then, by a theorem of Krasner in the cited reference, the sets \( \bigcup_{W \in C_\delta} W, \bigcup_{W \in C^\#_\delta} W \) are quasi-connected, for any nonnegative \( \delta \) (note that the empty set is trivially a quasi-connected domain). Our desired result then follows from the observations that these latter unions are the sets \( W_\delta(f) \) and \( W^\#_\delta(f) \), respectively.

Definition 1.4. Let \( V \) be a subset of \( \mathbb{R}^n \), \( j \) a positive integer, \( 1 \leq j \leq n \), and \( (a_1, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1} \). The symbol \( V^{(j)}(a_1, \ldots, a_{n-1}) \) denotes the subset of \( \mathbb{R} \) defined by
\[
V^{(j)}(a_1, \ldots, a_{n-1}) = \{ x \in \mathbb{R} : (a_1, \ldots, a_{j-1}, x, a_j, \ldots, a_{n-1}) \in V \}.
\]
If $V$ has the property that, for each integer $j$, $1 \leq j \leq n$, and for each $(n - 1)$-tuple $(a_1, \cdots, a_{n-1}) \in \mathbb{R}^{n-1}$, the set $V^{(j)}(a_1, \cdots, a_{n-1})$ is a quasi-connected domain, the set $V$ is said to be \textit{axially quasi-connected}.

**Corollary 1.5.** For $R(X_1, X_2, \cdots, X_n) \in \mathcal{O}[X_1, X_2, \cdots, X_n]$ let $W = \{ (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n : \text{ord } R(\xi_1, \xi_2, \cdots, \xi_n) = 0 \}$. Then the set $W$ is axially quasi-connected.

**Proof.** Let $j$ be any integer between 1 and $n$, and let $R^*_j(x) \in \mathcal{O}[x]$ be defined by

$$R^*_j(x) = R(a_1, \cdots, a_{j-1}, x, a_j, \cdots, a_{n-1}),$$

where $a_1, \cdots, a_{n-1}$ are arbitrarily chosen elements of $\mathcal{O}$; then $W^{(j)}(a_1, \cdots, a_{n-1})$ is either empty or equal to $W^*_j(R^*_j)$, and the preceding proposition applies.

2. **Uniqueness theorem.** In this section, a uniqueness theorem for analytic elements in several variables, generalizing the one-variable theory of Krasner, is proved. We do not claim to have a completely satisfactory generalization of Krasner's concept of a quasi-connected domain; in particular, while it is not sufficient only to assume that a subset of $\mathbb{R}^n$ be axially quasi-connected, it seems as though our definition of $W$ in the statement of the theorem is overly restrictive. However, it is only regions so defined with which we will be concerned in [4].

It is necessary to introduce some new ideas before the uniqueness theorem is stated.

**Definition 2.1.** Let $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$, $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ be a pair of elements of $\mathbb{R}^n$; we say that $\xi$ is \textit{directly axially joined} to $\eta$ if $\xi_i = \eta_i$ for all but possibly one of the indices $i = 1, 2, \cdots, n$. If $U$ is a subset of $\mathbb{R}^n$, and if $\xi, \eta$ are elements of $U$, we say that $\xi$ and $\eta$ are $U$-\textit{axially joined} if there is a sequence $\eta = \xi^{(0)}, \xi^{(1)}, \cdots, \xi^{(N)} = \xi$ with the property that, for $i = 0, 1, 2, \cdots, N$, $\xi^{(i)} \in U$, and, for $i = 1, 2, \cdots, N$, $\xi^{(i-1)}$ is directly axially joined to $\xi^{(i)}$.

It is clear from the definition that "is $U$-axially joined to" is an equivalence relation.

**Definition 2.2.** For $U \subseteq W \subseteq \mathbb{R}^n$, we define the \textit{axial join of $U$ in $W$}, $W'$, by

$$W' = \{ \xi \in W : \xi \text{ is } W\text{-axially joined to an element of } U \}.$$ 

**Proposition 2.3.** If $R(X_1, X_2, \cdots, X_n) \in \mathcal{O}[X_1, X_2, \cdots, X_n]$, ord $R(0, 0, \cdots, 0) = 0$, let $W = \{ (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n : \text{ord } R(\xi_1, \cdots, \xi_n) = 0 \}$ and let $\rho_1, \rho_2, \cdots, \rho_n$ be a set of positive numbers such that $U = \Gamma_{\rho_1} \times \Gamma_{\rho_2} \times \cdots \times \Gamma_{\rho_n}$ is not empty. Then, if $W'$ denotes the axial join of $U$ in $W$, $W' = W$. 
Proof. If \( n = 1 \), any two elements of \( \mathcal{R} \) are directly axially joined, and so \( W' = W \) trivially.

Assume the validity of the proposition for polynomials in \( n - 1 \) variables with coefficients in \( \mathbb{O} \), \( n \geq 2 \), and let \((\psi_1, \ldots, \psi_n) \in W\). We shall construct an element \((\eta_1, \eta_2, \ldots, \eta_n) \in U\) which is \( W \)-axially joined to \((\psi_1, \psi_2, \ldots, \psi_n)\).

Consider the image \( \tilde{R}(x_1, \ldots, x_n) \) of \( R(x_1, \ldots, x_n) \) under the residue class map: it follows from the definition of \( W \) that, if \( \tilde{\xi} \) denotes the residue class of \( \xi \) under reduction mod \( \mathbb{P} \), \((\xi_1, \ldots, \xi_n)\) is an element of \( W \) if and only if \( \tilde{R}(\tilde{\xi}_1, \ldots, \tilde{\xi}_n) \neq 0 \). Let the polynomials \( \tilde{R}', \tilde{R}_0 \) in \( k[X_n] \) be defined by

\[
\tilde{R}_0(x_n) = \tilde{R}(0, 0, \ldots, 0, x_n), \quad \tilde{R}'(x_n) = \tilde{R}(0, 0, \ldots, 0, x_n);
\]

since \( \tilde{R}_0(0)\tilde{R}'(\psi_n) \neq 0 \), the product of these two polynomials is not the zero polynomial. But \( k \) is infinite, so the existence of an element \( \eta \) of \( \mathbb{O} \) with the property \( \tilde{R}(\tilde{\eta})\tilde{R}_0(\tilde{\eta}) \neq 0 \) is guaranteed.

Let \( R^*(x_1, x_2, \ldots, x_{n-1}) = R(x_1, x_2, \ldots, x_{n-1}, \eta) \) and put \( W^* = \{(\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{O}^{n-1}; \text{ord } R^*(\xi_1, \xi_2, \ldots, \xi_{n-1}) = 0\}; \) then \( \text{ord } R^*(0, 0, \ldots, 0) = 0 \) and \((\psi_1, \psi_2, \ldots, \psi_{n-1}) \in W^* \), and therefore, by the induction hypothesis, \((\psi_1, \psi_2, \ldots, \psi_{n-1})\) is \( W^* \)-axially joined to an element \((\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta) \in \Gamma_{\rho_1} \times \Gamma_{\rho_2} \times \cdots \times \Gamma_{\rho_{n-1}} \). Thus, if we choose any element \( \eta_n \) of \( \Gamma_{\rho_n} \), the conclusion follows from the fact that \((\psi_1, \psi_2, \ldots, \psi_{n-1}, \eta)\) and \((\psi_1, \psi_2, \ldots, \psi_n)\) are directly axially joined, \((\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta)\) and \((\eta_1, \eta_2, \ldots, \eta_n)\) are axially joined, and \((\xi_1, \xi_2, \ldots, \xi_{n-1}) \in W^* \) if and only if \((\xi_1, \xi_2, \ldots, \xi_{n-1}, \eta) \in W\).

Remark 2.4. If \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \), \( \eta = (\eta_1, \eta_2, \ldots, \eta_n) \) are elements of \( \mathbb{O}^n \), and if \( \tilde{\xi}_i = \tilde{\eta}_i \), \( i = 1, 2, \ldots, n \), then \( \xi \in W \) if and only if \( \eta \in W \).

Theorem 2.5. For \( R(x_1, x_2, \ldots, x_n) \in \mathbb{O}[x_1, x_2, \ldots, x_n], \)

\( R(0, 0, \ldots, 0) \in \Gamma_0 \), let \( W = \{(\xi_1, \ldots, \xi_n) \in \mathbb{O}^n; \text{ord } R(\xi_1, \ldots, \xi_n) = 0\} \), and let \( \{f_m(x_1, x_2, \ldots, x_n), g_m(x_1, x_2, \ldots, x_n)\}, m = 1, 2, 3, \ldots \), be sequences of rational functions defined on \( W \) and converging uniformly to functions \( f(x_1, x_2, \ldots, x_n) \) and \( g(x_1, x_2, \ldots, x_n) \), respectively, on \( W \). Suppose, for some set of positive numbers \( \rho_1, \rho_2, \ldots, \rho_n \), the set \( U = \Gamma_{\rho_1} \times \Gamma_{\rho_2} \times \cdots \times \Gamma_{\rho_n} \) is not empty and \( f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \) on \( U \). Then \( f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \) identically on \( W \).

Proof. Let \( \xi \) be any element of \( W \). Then, by Proposition 2.3, there is an element \( \eta \) of \( U \) and a sequence \( \eta = \xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(N)} = \xi \) of elements of \( W \) such that adjacent members of the sequence are directly axially joined. The theorem will follow from construction of a sequence
Theorem 2.1 states that \( \Xi^{(0)} \), \( \Xi^{(1)} \), \( \cdots \), \( \Xi^{(N)} \) of sets, each of which satisfies the conditions:

1. \( \Xi^{(i)} \subseteq \mathcal{W} \).
2. For any choice of \( j, 1 \leq j \leq n \), and any element \( \xi = (\xi_1, \xi_2, \cdots, \xi_n) \) of \( \Xi^{(i)} \), the set
   \[ \Xi_{j,i}^{(i)} = \{ (\xi_1, \xi_2, \cdots, \xi_{j-1}, \xi_j, \xi_{j+1}, \cdots, \xi_n) \in \Xi^{(i)} \} \]
   has \( \xi_j \) as a limit point.
3. \( \xi^{(0)} \in \Xi^{(0)} \).
4. If \( (\xi_1, \cdots, \xi_n) \in \Xi^{(i)} \), then \( f(\xi_1, \cdots, \xi_n) = g(\xi_1, \cdots, \xi_n) \).

Such a sequence is constructed inductively. If the initial member \( \Xi^{(0)} \) is set equal to \( U \), it follows from the hypothesis that \( \Xi^{(0)} \) satisfies the four conditions.

Now, let \( 0 \leq i < N \), and suppose the set \( \Xi^{(i)} \) has been chosen in such a manner that conditions (1)--(4) are satisfied. Let

\[ \Xi^{(i+1)} = \bigcup_{j=1}^{n} \{ (\xi_1, \xi_2, \cdots, \xi_{j-1}, \eta_j, \xi_{j+1}, \cdots, \xi_n) \in \mathcal{W} : \]

\[ \text{for some } \xi_j, (\xi_1, \cdots, \xi_j, \cdots, \xi_n) \in \Xi^{(i)}. \]

It is obvious that \( \Xi^{(i+1)} \) so defined satisfies the first of our conditions.

Suppose \( \psi = (\psi_1, \cdots, \psi_n) \in \Xi^{(i+1)} \); if \( j \) is any integer, \( 1 \leq j \leq n \), we must show that \( \Xi^{(i+1)} \) has \( \psi_j \) as a limit point. But \( \psi \in \Xi^{(i+1)} \) implies the existence of an integer \( j' \), \( 1 \leq j' \leq n \), and an element \( \xi = (\xi_1, \cdots, \xi_n) \in \Xi^{(i)} \) such that \( \psi_i = \xi_i \) if \( i \neq j' \). If \( j' = j \), Remark 2.4 implies that all elements congruent to \( \psi_j \) mod \( \mathfrak{P} \) are in \( \Xi^{(i+1)} \), and so \( \psi_j \) is certainly a limit point of this latter set. On the other hand, if \( j' \neq j \), we use the fact that \( \xi_j \) is a limit point of \( \Xi^{(i)} \). Thus, we can choose an infinite subset \( \{ \xi_{j,l} \}, l = 0, 1, 2, \cdots, \) of \( \Xi^{(i)} \) such that \( \xi_{j,l} \rightarrow \xi_j \) as \( l \rightarrow \infty \), and such that these elements are all in the same residue class mod \( \mathfrak{P} \); but then, if we define \( \phi_i = (\phi_{i1}, \phi_{i2}, \cdots, \phi_{in}) \) by

\[ \phi_{i1} = \psi_j \text{ if } i = j', \]
\[ \psi_i \text{ if } i = j, \]
\[ \xi_i \text{ otherwise,} \]

Remark 2.4 implies that \( \phi_i \in \Xi^{(i+1)} \) for all \( l \), and, as \( l \rightarrow \infty \), \( \phi_{ij} \rightarrow \xi_j = \psi_j \), from which it follows that condition (2) is satisfied by the set \( \Xi^{(i+1)} \).

Condition (3) is fulfilled since \( \xi^{(i)} \in \Xi^{(i)} \) and \( \xi^{(i)}, \xi^{(i+1)} \) are directly axially joined.

Finally, if \( \psi = (\psi_1, \cdots, \psi_n) \in \Xi^{(i+1)} \), we choose \( \xi = (\xi_1, \cdots, \xi_n) \) and the integer \( j \) such that \( \psi_i = \xi_i \) if \( i \neq j \). Let

\[ R_{\psi,j}(X) = R(\psi_1, \psi_2, \cdots, \psi_{j-1}, X, \psi_{j+1}, \cdots, \psi_n), \]

and let \( W_{\psi,j} = \{ \eta_j : \text{ord } R_{\psi,j}(\eta_j) = 0 \} \). Proposition 2.3 tells us that \( W_{\psi,j} \) is a quasi-connected domain; but, by the induction hypothesis,
\( f(\xi_1, \cdots, \xi_{j-1}, X, \xi_{j+1}, \cdots, \xi_n) \) is identically equal to
\[
g(\xi_1, \cdots, \xi_{j-1}, X, \xi_{j+1}, \cdots, \xi_n)
\]
on the set \( \Xi_{\xi,j}^{(s)} \); since this subset of \( W_{v,j} \) has a limit point in itself, application of the one-variable uniqueness theorem proved by Krasner gives the result that
\[
f(\xi_1, \cdots, \xi_{j-1}, X, \xi_{j+1}, \cdots, \xi_n) = g(\xi_1, \cdots, \xi_{j-1}, X, \xi_{j+1}, \cdots, \xi_n)
\]
identically on \( W_{v,j} \). This proves our theorem.

REFERENCES


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