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ON TOTAL NONNORMING SUBSPACES

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ABSTRACT. A Banach space X has a total nonnorming subspace in its dual if and only if X has infinite codimension in its second dual.

Let X be a Banach space. A closed subspace M of X^* is said to be total if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$. The subspace M is said to be norming if $|||x||| = \sup\{|f(x)| : f \in M, \ \|f\| = 1\}$ is equivalent to the original norm on X. Clearly every norming subspace of X^* is total. The question "For which X does X^* have a total nonnorming subspace?" was raised by Dixmier [**2**]. It is well known that if X is quasi-reflexive (i.e., $\dim X^{**}/X < \infty$) then every total subspace of X^* is norming (cf. Petunin [**4**] and Singer [**7**]).

We prove here the converse, i.e., that for every non-quasi-reflexive space X there is a total nonnorming subspace in X^*. Strong partial results in this direction are already known (cf. Petunin [**4**]).

Let us first establish some definitions; we shall use in the following: A sequence (x_n) in a Banach space X is called a *basis* if for every $u \in X$ there is a unique sequence (a_n) of scalars such that $u = \sum a_n x_n$ (the series converging in norm). A sequence $(z_n) \subset X$ is a *basic sequence* if it is a basis for $[z_n]$ (= closed linear span of (z_n)). The basis constant for (z_n) is the smallest K such that $\|\sum_{i=1}^{m} a_i z_i\| \leqq K \|\sum_{i=1}^{m+n} a_i z_i\|$ for all (a_i), $m, n > 0$.

We begin with two lemmas, the first of which establishes the criterion we use in the main result.

LEMMA 1. *Let X be a Banach space. Assume that X has a closed infinite-dimensional subspace Y and that X^{**} has a closed infinite-dimensional subspace Z so that $Z \cap (Y^{**} + X) = \{0\}$. Then X^* contains a total nonnorming subspace.* (*We assume that X and Y^{**} are embedded canonically in X^{**}.*)

PROOF. There is no loss in generality to assume that Y is separable. Let $\{y_i^*\}_{i=1}^{\infty}$ be unit vectors in Y^* which are total over Y. Let $\{z_i\}_{i=1}^{\infty}$

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be a normalized basic sequence in Z. Define $T: Y^{**} \to Z$ by $Ty^{**} = \sum_{i=1}^{\infty} \epsilon_i y^{**}(y_i^*) z_i$ where $\epsilon_i > 0$, $\sum \epsilon_i \leqq 1/2$. Clearly T is compact, $\|T\| \leqq 1/2$ and $T_{|Y}$ is one to one. Let $U = \{y^{**} + Ty^{**}; y^{**} \in Y^{**}\}$. It is easy to verify that the unit cell of U and hence, by Krein's theorem, also U itself is w^* closed in X^{**}. Let M be the subspace $U^{\perp}$ of X^*. Since by our assumption $U \cap X = \{0\}$ it follows that M is total. M is however, nonnorming. Indeed, let $\epsilon > 0$, and let $y \in Y$ be such that $\|y\| = 1$ and $\|Ty\| \leqq \epsilon$. For every $f \in M$ we have $f(y + Ty) = 0$ and hence $|f(y)| = |f(Ty)| \leqq \epsilon \|f\|$.

LEMMA 2. *If X is not reflexive, X contains an infinite-dimensional subspace Y such that X/Y is not reflexive.*

PROOF. By Singer [**6**] and Pelczynski [**5**] $X \supset (x_n)$, a basic sequence with $\|x_n\| \geqq 1$ for all n such that $(\sum_1^p x_n)$ is bounded (in p). Let $Y = [x_{2n-1}]$ and φ be the quotient map of X onto X/Y. Then, if K is the basis constant for (x_n), it is well known that $\|\varphi(x_{2n})\| \geqq 1/2K$. Further, $(\varphi(x_{2n}))$ is basic and $\sum_1^p \varphi(x_{2n}) = \varphi(\sum_1^{2p} x_n)$ is bounded. A basic sequence such as $(\varphi(x_{2n}))$ cannot exist in a reflexive space [**6**], so X/Y is not reflexive.

THEOREM. *Let X be a Banach space with* $\dim X^{**}/X = \infty$. *Then X^* contains a total nonnorming subspace.*

PROOF. By Lemma 1, it suffices to find an infinite-dimensional subspace Y of X such that $\dim X^{**}/(X + Y^{**}) = \infty$. (It follows, for example, as in [**3**], that there is a subspace Z of X^{**} of infinite dimension such that $Z \cap (X + Y^{**}) = \{0\}$.) For this, we may as well also assume that X has no infinite-dimensional reflexive subspace (for such a subspace could take the role of Y in Lemma 1). Now, using Lemma 2, construct a chain of subspaces

$$X = X_1 \supset X_2 \supset X_3 \supset X_4 \supset \cdots$$

such that for each k, X_k/X_{k+1} is nonreflexive. Next, for $k = 1, 2, \cdots$, let $y_k \in X_k \sim X_{k+1}$ and set $Y = [y_k]$. To see that Y satisfies the hypothesis of Lemma 1, notice first that $Y^{**} = [y_1, y_2, \cdots, y_k] + (X_k \cap Y)^{**}$. Thus, for each k, $X + Y^{**} = X + (X_k \cap Y)^{**} \subset X + X_k^{**}$. By [**1**, Theorem 4.1], $X^{**}/(X + X_k^{**})$ is isomorphic to $((X/X_k)^{**}/(X/X_k))$ so that $X + Y^{**}$ has codimension $\geqq k$ for every k, proving the theorem.

REMARK. We use implicitly the fact [**1**, Theorem 4.1] that $X + Y^{**}$ is always closed in X^{**}. We have also used [**1**, Corollary 4.2] to see that $\dim ((X/X_k)^*/(X/X_k)) \geqq k$ for each k.

One interesting corollary of the theorem is the following stronger version of Lemma 2. "*If X is non-quasi-reflexive, then X contains an* *infinite-dimensional subspace Y such that X/Y is non-quasi-reflexive.*" If X contains an infinite-dimensional reflexive subspace R, then X/R is the desired factor. Otherwise, let Y be the subspace constructed in the proof of the theorem.

REFERENCES

1. P. Civin and B. Yood, *Quasi-reflexive spaces*, Proc. Amer. Math. Soc. **8** (1957), 906–911. MR **19**, 756.

2. J. Dixmier, *Sur un théorème de Banach*, Duke Math. J. **15** (1948), 1057–1071. MR **10**, 306.

3. V. I. Gurariĭ and M. I. Kadec, *Minimal systems and quasi-complements in Banach space*, Dokl. Akad. Nauk SSSR **145** (1962), 256–258 = Soviet Math. Dokl. **3** (1962), 966–968. MR **26** #6728.

4. Ju. I. Petunin, *Conjugate Banach spaces containing subspaces of zero characteristic*, Dokl. Akad. Nauk SSSR **154** (1964), 527–529 = Soviet Math. Dokl. **5** (1964), 131–133. MR **28** #2425.

5. A. A. Pełczyński, *A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces"*, Studia Math. **21** (1961/62), 371–374. MR **26** #4156.

6. I. Singer, *Basic sequences and reflexivity of Banach spaces*, Studia Math. **21** (1961/62), 351–369. MR **26** #4155.

7. ———, *On bases in quasi-reflexive Banach spaces*, Rev. Math. Pures Appl. (Bucarest) **8** (1963), 309–311. MR **31** #2599.

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