ON TOTAL NONNORMING SUBSPACES
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Abstract. A Banach space $X$ has a total nonnorming subspace in its dual if and only if $X$ has infinite codimension in its second dual.

Let $X$ be a Banach space. A closed subspace $M$ of $X^*$ is said to be total if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$. The subspace $M$ is said to be norming if $\|x\| = \sup \{|f(x)| : f \in M, \|f\| = 1\}$ is equivalent to the original norm on $X$. Clearly every norming subspace of $X^*$ is total. The question "For which $X$ does $X^*$ have a total nonnorming subspace?" was raised by Dixmier [2]. It is well known that if $X$ is quasi-reflexive (i.e., $\dim X^{**}/X < \infty$) then every total subspace of $X^*$ is norming (cf. Petunin [4] and Singer [7]).

We prove here the converse, i.e., that for every non-quasi-reflexive space $X$ there is a total nonnorming subspace in $X^*$. Strong partial results in this direction are already known (cf. Petunin [4]).

Let us first establish some definitions; we shall use in the following: A sequence $(x_n)$ in a Banach space $X$ is called a basis if for every $u \in X$ there is a unique sequence $(a_n)$ of scalars such that $u = \sum a_n x_n$ (the series converging in norm). A sequence $(z_n) \subset X$ is a basic sequence if it is a basis for $[z_n]$ (= closed linear span of $(z_n)$). The basis constant for $(z_n)$ is the smallest $K$ such that $\| \sum_{i=1}^{m+n} a_i z_i \| \leq K \| \sum_{i=1}^{m} a_i z_i \|$ for all $(a_i), m, n > 0$.

We begin with two lemmas, the first of which establishes the criterion we use in the main result.

Lemma 1. Let $X$ be a Banach space. Assume that $X$ has a closed infinite-dimensional subspace $Y$ and that $X^{**}$ has a closed infinite-dimensional subspace $Z$ so that $Z \cap (Y^{**} + X) = \{0\}$. Then $X^*$ contains a total nonnorming subspace. (We assume that $X$ and $Y^{**}$ are embedded canonically in $X^{***}$.)

Proof. There is no loss in generality to assume that $Y$ is separable. Let $\{y_i^{**}\}_{i=1}^{\infty}$ be unit vectors in $Y^*$ which are total over $Y$. Let $\{z_i\}_{i=1}^{\infty}$
be a normalized basic sequence in $Z$. Define $T: Y^{**} \to Z$ by $T_{y^{**}} = \sum_{i=1}^{\infty} \epsilon_i (y_i^{**}) z_i$ where $\epsilon_i > 0$, $\sum \epsilon_i \leq 1/2$. Clearly $T$ is compact, $\|T\| \leq 1/2$ and $T_{1_Y}$ is one to one. Let $U = \{y^{**} + Ty^{**}; y^{**} \in Y^{**}\}$. It is easy to verify that the unit cell of $U$ and hence, by Krein’s theorem, also $U$ itself is $w^*$ closed in $X^{**}$. Let $M$ be the subspace $U \perp$ of $X^*$. Since by our assumption $U \cap X = \{0\}$ it follows that $M$ is total. $M$ is however, nonnorming. Indeed, let $\epsilon > 0$, and let $y \in Y$ be such that $\|y\| = 1$ and $\|Ty\| \leq \epsilon$. For every $f \in M$ we have $f(y + Ty) = 0$ and hence $|f(y)| = |f(Ty)| \leq \epsilon \|f\|$. 

**Lemma 2.** If $X$ is not reflexive, $X$ contains an infinite-dimensional subspace $Y$ such that $X|Y$ is not reflexive.

**Proof.** By Singer [6] and Pelczynski [5] $X \supset (x_n)$, a basic sequence with $\|x_n\| \geq 1$ for all $n$ such that $(\sum_1^\infty x_n)$ is bounded (in $p$). Let $Y = [x_{2n-1}]$ and $\varphi$ be the quotient map of $X$ onto $X/Y$. Then, if $K$ is the basis constant for $(x_n)$, it is well known that $\|\varphi(x_{2n})\| \geq 1/2K$. Further, $(\varphi(x_{2n}))$ is basic and $\sum_1^\infty \varphi(x_{2n}) = \varphi(\sum_1^\infty x_n)$ is bounded. A basic sequence such as $(\varphi(x_{2n}))$ cannot exist in a reflexive space [6], so $X/Y$ is not reflexive.

**Theorem.** Let $X$ be a Banach space with $\dim X^{**}/X = \infty$. Then $X^*$ contains a total nonnorming subspace.

**Proof.** By Lemma 1, it suffices to find an infinite-dimensional subspace $Y$ of $X$ such that $\dim X^{**}/(X + Y^{**}) = \infty$. (It follows, for example, as in [3], that there is a subspace $Z$ of $X^{**}$ of infinite dimension such that $Z \cap (X + Y^{**}) = \{0\}$.) For this, we may as well also assume that $X$ has no infinite-dimensional reflexive subspace (for such a subspace could take the role of $Y$ in Lemma 1). Now, using Lemma 2, construct a chain of subspaces

$$X = X_1 \supset X_2 \supset X_3 \supset X_4 \supset \cdots$$

such that for each $k$, $X_k/X_{k+1}$ is nonreflexive. Next, for $k = 1, 2, \ldots$, let $y_k \in X_k \sim X_{k+1}$ and set $Y = [y_k]$. To see that $Y$ satisfies the hypothesis of Lemma 1, notice first that $Y^{**} = (y_1, y_2, \ldots, y_k) + (X_k \cap Y)^{**}$. Thus, for each $k$, $X + Y^{**} = X + (X_k \cap Y)^{**} \subset X + X_k^{**}$. By [1, Theorem 4.1], $X^{**}/(X + X_k^{**})$ is isomorphic to $((X/X_k)^{**}/(X/X_k))$ so that $X + Y^{**}$ has codimension $\geq k$ for every $k$, proving the theorem.

**Remark.** We use implicitly the fact [1, Theorem 4.1] that $X + Y^{**}$ is always closed in $X^{**}$. We have also used [1, Corollary 4.2] to see that

$$\dim ((X/X_k)^{**}/(X/X_k)) \geq k$$

for each $k$.

One interesting corollary of the theorem is the following stronger version of Lemma 2. "If $X$ is non-quasi-reflexive, then $X$ contains an
infinite-dimensional subspace $Y$ such that $X/Y$ is non-quasi-reflexive.” If $X$ contains an infinite-dimensional reflexive subspace $R$, then $X/R$ is the desired factor. Otherwise, let $Y$ be the subspace constructed in the proof of the theorem.

REFERENCES


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