THE WALLMAN COMPACTIFICATION IS AN EPIREFLECTION

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Abstract. It is shown that a map having an extension to a closed map between the Wallman compactifications of its domain and range has a unique such extension. A consequence is that the collection of such maps forms the morphisms of a category on which the Wallman compactification is an epireflection, answering a question raised by Herrlich.

We establish the existence of a category $C$ of spaces and maps on which the Wallman compactification is an epireflection functor. This answers a question raised by Herrlich in [1] as to whether such a category exists.

A space is a $T_1$ topological space and a map is a continuous function. For categorical terminology see [1].

The class of objects of $C$ will be the class of spaces, and the morphisms will be the class of maps $f : X \to Y$ such that there is a closed map $g : wX \to wY$ with $gwX = wY f$, where $wX, wY$ are the inclusions of $X, Y$ into their Wallman compactifications $wX, wY$. Any such closed map $g$ is called a $w$-extension of $f$.

It is clear that for each $X \in C$ the maps $1_X$ and $w_X$ are morphisms, since $1_{wX}$ is a $w$-extension of these maps. Also trivially if $f$ is a morphism and $g$ is a $w$-extension of $f$ then $g$ is a morphism (being its own $w$-extension). Finally since the composition of closed maps is a closed map it follows that the composition of morphisms is a morphism.

To show that the Wallman compactification is an epireflection on this category we need only show that each morphism has a unique $w$-extension. To do this we examine the construction of the Wallman compactification of $X$ as given for example in [2].

The points of the Wallman compactification are the maximal closed filters $[p]$ on $X$; we write $[p]$ for the filter and $p$ for the point of $wX$. For each $p \in wX$ there is the open filter $\langle p \rangle$ consisting of the open subsets of $Y$ which contain a member of $[p]$. The neighborhood filter of $p \in wX$ is

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265
generated by the sets $V^* = \{ q \in wX : V \in \langle q \rangle \}$ taken over all $V \in \langle p \rangle$. Since for each $x \in X$ the principal filter $[x]$ is maximal closed we have the function $w_x$, and it is an embedding of $X$ into $wX$.

**Lemma.** If $f : X \rightarrow Y$ is a morphism and $g : wX \rightarrow wY$ is a $w$-extension of $f$ then for any open $U \subseteq Y$ we have $U \in \langle g(p) \rangle$ if and only if there is $C \in [p]$ with $cl_Y f [C] \subseteq U$.

**Proof.** Suppose $C \in [p]$. Then $p \in cl_{wX}w_X [C]$ and so

$$g(p) \in g[cl_{wX}w_X [C]] \subseteq cl_{wY}g_{wX} [C] \subseteq cl_{wY}w_Y f [C];$$

therefore $cl_Y f [C] \in [g(p)]$. Thus if $cl_Y f [C] \subseteq U$ for some $C \in [p]$ and open $U \subseteq Y$ then $B \subseteq U$ for some $B \in [g(p)]$, and thus $U \in \langle g(p) \rangle$.

Conversely suppose $B \subseteq U$ for some $B \in [g(p)]$ and open $U \subseteq Y$. Then $g^-[U^*]$ is a neighborhood of $p$, so there is $W \in \langle p \rangle$ with $W^* \subseteq g^-[U^*]$. Since $W \in \langle p \rangle$ there is $C \in [p]$ with $C \subseteq W$. Now $cl_{wX}w_X [C] = C^* \subseteq W^*$, and since $g$ is closed we have $cl_{wY}w_Y f [C] = cl_{wY}g_{wX} [C] = cl_{wY}g[cl_{wX}w_X [C]] = g[C^*] \subseteq g[W^*] \subseteq U^*$; therefore $cl_Y f [C] \subseteq U$.

Thus if $U \in \langle g(p) \rangle$ there is $C \in [p]$ with $cl_Y f [C] \subseteq U$.

In accordance with the Lemma the filter $\langle g(p) \rangle$ and hence the point $g(p)$ is entirely determined by the map $f$, hence the following holds:

**Corollary 1.** Each morphism has a unique $w$-extension.

Now suppose $g$ and $h$ are morphisms with $gw_X = hw_X$. Then $g$ and $h$ have $w$-extensions $m$ and $n$. Since $nw_X = w_Y gw_X = w_Y hw_X = mw_X$ then by the preceding corollary $n = m$ and therefore $w_Y h = n = m = wy g$, from which we find $h = g$. We therefore find the further result:

**Corollary 2.** Each $w_X$ is an epimorphism.

Summing up the preceding results, and making the trivial observation that each morphism with compact domain and range is a closed map we find:

**Theorem.** The category of closed maps and compact spaces is an epireflective subcategory of the category $C$, and the epireflection of a space is its Wallman compactification.

It is known that every closed onto map has a closed onto $w$-extension; see Ponomarev [3] for this result, first shown by Arhangel'skii. Thus each closed onto map is a morphism. Another class of morphisms is the class of maps with compact Hausdorff range; as is well known (and readily established using extension by regularity) each such map has an extension to the Wallman compactification of its domain, and the extension must be closed.
REFERENCES


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