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CONSTRUCTING SEQUENCES OF DIVIDED POWERS[1]

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ABSTRACT. In my *Sequences of divided powers in irreducible, cocommutative Hopf algebras*, I demonstrated the existence of extensions of sequences of divided powers over arbitrary fields, if certain coheight conditions are met. Here, I show that if the characteristic of the field does not divide n, every sequence of divided powers of length $n-1$, in a cocommutative Hopf algebra, has an extension that can be written as a polynomial in the previous terms. (An algorithm for finding these polynomials is given, together with a list of some of them.) Furthermore, I show that if one uses this method successively for constructing a sequence of divided powers over a primitive, the only obstructions will occur at powers of the characteristic of the field.

Some of the basic definitions of this paper are the following:

(1) If H is a Hopf algebra and $0 \neq g \in H$, then g is a grouplike if $\Delta g = g \otimes g$.

(2) If $h \in H$, then h is a primitive if $\Delta h = h \otimes 1 + 1 \otimes h$.

(3) A Hopf algebra will be called irreducible if every nontrivial subcoalgebra contains a fixed, nontrivial subcoalgebra, i.e., if H is irreducible, the identity is the unique grouplike.

(4) An irreducible Hopf algebra will be called graded, if there exists a set of subspaces $\{H_i\}_{i=0}^{\infty}$ such that

(a) $H = \bigoplus_{i=0}^{\infty} H_i$;

(b) $H_0 = 1 \cdot k$;

(c) $\Delta H_i \subset \bigoplus_{j=0}^{i} H_j \otimes H_{i-j}$;

(d) $H_i \cdot H_j \subset H_{i+j}$.

(5) A sequence of divided powers ${}^0x = 1, {}^1x, {}^2x, \cdots, {}^nx$ is a set of elements in a cocommutative, irreducible Hopf algebra such that $\Delta\, {}^tx = \sum_{i=0}^{t} {}^ix \otimes {}^{t-i}x$, for all $0 \leqq t \leqq n$.

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(6) ${}^{n+1}x$ will be called an extension of the sequences of divided powers ${}^0x = 1, {}^1x, {}^2x, \cdots, {}^nx$, if ${}^0x = 1, {}^1x, {}^2x, \cdots, {}^nx, {}^{n+1}x$ is also a sequence of divided powers.

FACT. An irreducible cocommutative bialgebra is a Hopf algebra.

PROOF. Proposition 9.2.5, p. 196 of [3].

NOTATION. Throughout the remainder of this paper, H will be an irreducible, cocommutative, commutative Hopf algebra over a field k. H will have argumentation ϵ, diagonalization Δ, and identity 1. All tensor products will be over k.

Let $J_n \equiv k[X_1, X_2, \cdots, X_n]$ and make it into a graded Hopf algebra via $\Delta X_i = \sum_{j=0}^{i} X_j \otimes X_{i-j}$ (where $X_0 \equiv 1$), and $\epsilon(X_j) = 0$ for $1 \leqq i \leqq n$, and $\deg X_i = i$ for $0 \leqq i \leqq n$.

Note that J_n is an irreducible, cocommutative, commutative Hopf algebra with $1, X_1, X_2, \cdots, X_n$ forming a sequence of divided powers.

DEFINITION. Let $M = \prod X_i^{t_i}$ be a monomial in J_n. The largest i such that $t_i \neq 0$ will be called the index of M.

LEMMA 1. *In J_n, the only monomial with index $\leqq r$ that contains $T = (\prod_{i=1}^{r-1} X_i^{t_i}) \otimes X_r^{t_r}$, $t_r > 0$, in its diagonalization with a nonzero coefficient is $M = \prod_{i=1}^{r} X_i^{t_i}$.*

PROOF. It is clear that no monomial with index less than r can contain T in its diagonalization. Assume $N = \prod_{i=1}^{r} X_i^{u_i}$ contains T in its diagonalization. Now $\Delta N = \prod_{i=1}^{r} (\sum_{j=0}^{i} X_j \otimes X_{i-j})^{u_i}$ (where $X_0 \equiv 1$) and, therefore, the only term in the expansion of ΔN with only X_r's on the right-hand side and no X_r's on the left-hand side is $(\prod_{i=1}^{r-1} X_i^{u_i}) \otimes X_r^{u_r}$. Therefore $u_i = t_i$, $i = 1, 2, \cdots, r$, and $N = M$.

THEOREM 2. *In J_{n-1} over Q (the rationals), the sequence of divided powers $1, X_1, X_2, \cdots, X_{n-1}$ can be extended uniquely by a homogeneous polynomial, $P_n(X)$, of degree n. The coefficients of this polynomial will be in $Z[1/n]$. In fact, if M is a monomial in $P_n(X)$ with index r, its coefficient will be in $Z[1/p_1p_2 \cdots p_s]$ where the p_i are the prime factors of n such that $n/r \geqq p_i$. In particular, if r is larger than the largest nontrivial divisor of n, its coefficient will be in Z.*

PROOF. By Corollary 13.0.3, p. 278 of [3], any irreducible, cocommutative Hopf algebra H over a field of characteristic 0 is generated as an algebra by its primitives. Assume H is graded. Then each homogeneous component of each primitive is a primitive and we can say that H is generated by homogeneous primitives. Now consider J_n. It is generated by homogeneous primitives and, therefore, there must exist a homogeneous primitive containing X_n as one of its terms. This primitive must, perforce, be of the form: $X_n - P_n(X_1, X_2, \cdots, X_{n-1})$. Since X_n is an extension of

$1, X_1, X_2, \cdots, X_{n-1}$, and $X_n - P_n(X_1, X_2, \cdots, X_{n-1})$ is a primitive it follows that $P_n(X_1, X_2, \cdots, X_{n-1})$ will be the desired extension in J_{n-1}.

We now investigate the coefficients of $P_n(X)$. First, we show that the coefficient of $X_i^{n/i}$ in $P_n(X)$ (for $i \mid n$) will be $(-1)^{n/i}(i/n)$.

Since $P_n(X)$ is an nth divided power of $1, X_1, X_2, \cdots, X_{n-1}$, $\Delta P_n(X)$ contains the term $X_i \otimes X_{n-i}$. A little thought will make it evident that the factors of a monomial containing $X_i \otimes X_{n-i}$ in its diagonalization must be X_i and X_{n-i}, i.e., the unique monomial containing $X_i \otimes X_{n-i}$ in its diagonalization is $X_i X_{n-i}$. Therefore, the coefficient of $X_i X_{n-i}$ is 1.

Now, the diagonalization of $X_i X_{n-i}$ contains the term $X_i^2 \otimes X_{n-2i}$. Since this term does not appear in $\Delta P_n(X)$, it must be cancelled by terms in the diagonalization of other monomials in $P_n(X)$. Again, a little thought will show that for a monomial to contain $X_i^2 \otimes X_{n-2i}$ in its diagonalization, it must contain as factors, combinations of X_i, X_{n-2i} and X_{n-i}. By checking all the possibilities, one can see that the only monomial other than $X_i X_{n-i}$ that contains $X_i^2 \otimes X_{n-2i}$ in its diagonalization is $X_i^2 X_{n-2i}$. Therefore, the coefficient of $X_i^2 X_{n-2i}$ is -1. Now, $\Delta(-X_i^2 X_{n-2i})$ contains the term $-X_i^3 \otimes X_{n-3i}$ and in a similar way to the above, we can show that the coefficient of $X_i^3 X_{n-3i}$ is $+1$.

Continuing in this manner, we find that the coefficient of $X_i^{(n/i)-2} X_{2i}$ is $(-1)^{(n/i)-1}$. Since its diagonalization contains the term $X_i^{(n/i)-1} \otimes X_i$, the coefficient of $X_i^{n/i}$ (which contains the term $(n/i) X_i^{(n/i)-1} \otimes X_i$ in its diagonalization) must be $(-1)^{n/i}\ (i/n)$.

To complete the proof, we show that the coefficient in $P_n(X)$ of $M \equiv \prod_{i=1}^{r} X_i^{t_i}$ ($t_r \neq 0$ and some $t_i \neq 0$, $1 \leqq i \leqq r$), a monomial of degree n, is an integral combination of the coefficients in $P_n(X)$ of larger monomials. (We say one monomial is "larger" than another if its index is larger.) We will do this by (1) finding a term in ΔM, other than $M \otimes 1$ or $1 \otimes M$, with coefficient one that occurs elsewhere only in the diagonalization of larger monomials. (2) Since the coefficient of every term in $\Delta P_n(X) - 1 \otimes P_n(X) - P_n(X) \otimes 1$ is either zero or one, (3) descending induction (on the index of the monomial) will complete the proof.

(1) A term in ΔM is $T \equiv (\prod_{i=1}^{r-1} X_i^{t_i}) \otimes X_r^{t_r}$ and one can check that it has coefficient one. By Lemma 1, every monomial other than M, that contains T in its diagonalization, must be larger than M.

(2) So let $\{M_i\}_{i \in I}$ be the set of monomials larger than M in J_{n-1}, let b_i be the coefficient of T in ΔM_i, and let c_i be the coefficient of M_i in $P_n(X)$. Then $e = c + \sum_{i \in I} c_i b_i$ where e is the coefficient of T in $\Delta P_n(X)$ (either zero or one) and c is the coefficient of M in $P_n(X)$.

(3) Now all the c_i's are in $Z[1/p_1 p_2 \cdots p_s]$ where the p_i's are the prime factors of n such that $n/r \geqq p_i$. (If M_i is of the form $X_j^{n/j}$ we proved this statement above, and if M_i is not of this form it is true by the induction

hypothesis. Note that we showed, en passant, that the coefficient of the largest monomial, $X_{n-1}X_1$, is one.) Therefore, since all the b_i's are integers, it follows that c is in $Z[1/p_1p_2\cdots p_s]$.

Uniqueness is clear, since we have found the unique possible coefficient of each monomial in $P_n(X)$. Q.E.D.

REMARK. We give here an algorithm to find the P_n's as defined in the previous theorem. The algorithm is derived from methods used in the proof of the theorem and is not specifically proved.

In P_n:

(a) the coefficient of $X_i^{n/i}$ is $(-1)^{n/i}(i/n)$ $(1 \leqq i < n$ and $i \mid n)$;

(b) the coefficient of X_iX_{n-i} is 1 $(1 \leqq i < n/2)$;

(c) the coefficient of $M = \prod_{i=1}^{r} X_i^{t_i}$ $(t_r \neq 0$, M not of the form (a) or (b)) will be minus the sum of the coefficients of monomials of the form: $\prod_{i=1}^{s} X_i^{u_i}$ with $u_s \neq 0$, $2r > s > r$, $u_i + u_{r+i} = t_i$ $(1 \leqq i < r)$, and $\sum_{i=r}^{s} u_i = t_r$.

In Table I, I have listed the first seven polynomials (of degree 2–8). It is interesting to empirically observe (though I can not explain it), that the coefficient of a term is positive, if the sum of its exponents is even and vice versa. (I have found this to be true for all the P_n's I have computed, up to $n = 20$.)

TABLE 1

The polynomial of degree n $(n = 2, \cdots, 8)$ which will extend the sequence of divided powers $1, X_1, X_2, \cdots, X_{n-1}$ in J_{n-1}.

$n = 2$ $\quad \frac{1}{2}X_1^2.$

$n = 3$ $\quad X_2X_1 - \frac{1}{3}X_1^3.$

$n = 4$ $\quad X_3X_1 + \frac{1}{2}X_2^2 - X_2X_1^2 + \frac{1}{4}X_1^4.$

$n = 5$ $\quad X_4X_1 + X_3X_2 - X_3X_1^2 - X_2^2X_1 + X_2X_1^3 - \frac{1}{5}X_1^5.$

$n = 6$ $\quad X_5X_1 + X_4X_2 - X_4X_1^2 + \frac{1}{2}X_3^2 - 2X_3X_2X_1 + X_3X_1^3 - \frac{1}{3}X_2^3 + 1\frac{1}{2}X_2^2X_1^2 - X_2X_1^4 + \frac{1}{6}X_1^6.$

$n = 7$ $\quad X_6X_1 + X_5X_2 - X_5X_1^2 + X_4X_2 - 2X_4X_2X_1 + X_4X_1^3 - X_3^2X_1 - X_3X_2^2 + 3X_3X_2X_1^2 - X_3X_1^4 + X_2^3X_1 - 2X_2^2X_1^3 + X_2X_1^5 - \frac{1}{7}X_1^7.$

$n = 8$ $\quad X_7X_1 + X_6X_2 - X_6X_1^2 + X_5X_3 - 2X_5X_2X_1 + X_5X_1^3 + \frac{1}{2}X_4^2 - 2X_4X_3X_1 - X_4X_2^2 + 3X_4X_2X_1^2 - X_4X_1^4 - X_3^2X_2 + 1\frac{1}{2}X_3^2X_1^2 + 3X_3X_2^2X_1 - 4X_3X_2X_1^3 + X_3X_1^5 + \frac{1}{4}X_2^4 - 2X_2^3X_1^2 + 2\frac{1}{2}X_2^2X_1^4 - X_2X_1^6 + \frac{1}{8}X_1^8.$

COROLLARY 3. *Let* char $k = a$ *(a either* 0 *or positive). If* ${}^0x = 1$, ${}^1x, \cdots, {}^{n-1}x$ *is a sequence of divided powers in H, then the sequence can be extended by a polynomial in* ${}^1x, {}^2x, \cdots, {}^{n-1}x$ *if* $a \nmid n$ *or if* $a = 0$.

PROOF. Consider J_{n-1} over k. Define a Hopf algebra map χ between J_{n-1} and H via $X_i \to {}^ix$. By the theorem the coefficients of $P_n(X)$ are in $Z[1/n]$ and therefore $P_n(X)$ can be written in J_{n-1}. The image of $P_n(X)$ under χ will be the desired extension of $1, {}^1x, \cdots, {}^{n-1}x$. Q.E.D.

Note that if $a > 0$, the coheight (in the sense of [1]) of the extension will be the same as the coheight of ${}^nx$.

COROLLARY 4. *Assume* char $k = p > 0$. *If* ${}^0x = 1, {}^1x, \cdots, {}^mx$ *and* ${}^0y = 1, {}^1y, \cdots, {}^ny$ *are* 2 *sequences of divided powers such that* ${}^ix = {}^iy$, $0 \leqq i \leqq t$, *and if* ${}^0x, {}^1x, \cdots, {}^mx$ *can be extended to* ${}^{tp}x$, *then* ${}^0y, {}^1y, \cdots, {}^ny$ *can be extended to* ${}^{tp}y$ *by polynomials in* ${}^1x, {}^2x, \cdots, {}^{tp}x, {}^1y, {}^2y, \cdots, {}^ny$.

PROOF. Assume inductively that we have extended ${}^0y, {}^1y, \cdots, {}^ny$ to ${}^0y, {}^1y, \cdots, {}^{u-1}y$, $n < u \leqq tp$, using polynomials in ${}^1x, {}^2x, \cdots, {}^{tp}x, {}^1y, {}^2y, \cdots, {}^ny$. If $p \nmid u$, then we can extend again using Corollary 3. So assume $u = rp$, $r \leqq t$. Let $J = k[X_1, X_2, \cdots, X_{rp}, Y_{t+1}, Y_{t+2}, \cdots, Y_{rp-1}]$ where $\epsilon(X_i) = \epsilon(Y_i) = 0$ $(i \geqq 1)$, $\Delta X_i = \sum_{j=1}^{i} X_j \otimes X_{i-j}$, and $\Delta Y_i = \sum_{j=0}^{i} Y_i \otimes Y_{i-j}$ (letting $Y_0 = X_0 = 1$ and $Y_i = X_i$ if $i \leqq t$). Consider $P_{rp}(Y) - P_{rp}(X) + X_{rp}$. Since $Y_i = X_i$ $(i \leqq t)$ all monomials have index greater than t and $t \geqq r$. According to the theorem, the coefficients of such monomials in $P_{rp}(X)$ are in $Z[1/p_1p_2 \cdots p_s]$ where the p_i's are the prime factors of u less than p. Therefore, we can write $P_{rp}(Y) - P_{rp}(X) + X_{rp}$ in J. Since $X_{rp} - P_{rp}(X)$ is a primitive, $P_{rp}(Y) - P_{rp}(X) + X_{rp}$ will be an extension of $1, Y_1, Y_2, \cdots, Y_{rp-1}$ in J. Now map $J \to H$ via $X_i \to {}^ix$ and $Y_i \to {}^iy$. The image of $P_{rp}(Y) - P_{rp}(X) + X_{rp}$ will be the desired extension. Q.E.D.

We have shown that if H is an irreducible, cocommutative, commutative Hopf algebra over a field of characteristic $p > 0$, an extension to any sequence of divided powers of length $n - 1$ can be constructed if $p \nmid n$. If $p \mid n$, this is not, in general, true. However, if we start with a primitive, use the P_n's of Theorem 2 to construct a $p - 1$ length sequence, and if, in some manner, we can find a pth divided power, we can then use the P_n's to construct a $p^2 - 1$ length sequence. Further, if we can now find a p^2 divided power, the P_n's allow us to construct a $p^3 - 1$ sequence. This process may be continued indefinitely. In other words, obstructions can only arise at $p^{n\text{th}}$ divided powers, not at every np divided power. This is the import of Theorem 7 and Corollary 8, below. First, however, we need the following:

LEMMA 5. *Let n be any positive integer greater than one, and p any prime. Then there exists α, $1 \leqq \alpha < n$, such that $p \nmid \binom{n}{\alpha}$, if n is not a power of p. If n is a power of p, thère exists α, $1 \leqq \alpha < n$, such that $p^2 \nmid \binom{n}{\alpha}$.*

The proof is trivial.

DEFINITION 6. (a) Let p be a prime. If n is a power of p, $P_{n,p} \equiv X_n$. If n is not a power of P, define $P_{n,p}$ inductively by substituting $P_{i,p}$ for X_i $(i = 1, 2, \cdots, n-1)$ in P_n.

(b) Make $k[X_1, X_p, \cdots, X_{p^m}]$ into a graded Hopf algebra via $\Delta X_{p^i} = \sum_{j=0}^{p^i} P_{j,p} \otimes P_{p^i-j,p}$ (with $P_{0,p} \equiv 1$), $\epsilon(X_{p^i}) = 0$ and $\deg X_{p^i} = p^i$.

Note that $1; P_{1,p}; P_{2,p}; \cdots$ is a sequence of divided powers and that $P_{i,p}$ is homogeneous of degree i. The $k[X_1, X_p, \cdots, X_{p^m}]$ are isomorphic to the familiar Witt Hopf algebras. See Theorem 2.23, p. 70 of [**2**].

THEOREM 7. *For any n and any prime p, the coefficients of $P_{n,p}$ as a polynomial in $X_1, X_p, \cdots, X_{p^m}$ $(p^m \leqq n < p^{m+1})$ are in*

$$Z_p = Z[1/2, 1/3, 1/5, \cdots, \widehat{1/p}, \cdots].$$

PROOF. Assume inductively that the theorem is true for $P_{i,p}$, $i < n$, and that n is not a power of p (otherwise, the theorem is clear). Let χ be the natural map:

$$Z_p[X_1, X_p, \cdots, X_{p^m}] \rightarrow k[X_1, X_p, \cdots, X_{p^m}]$$

where k is any field of characteristic p. Since we have assumed that the $P_{i,p}$'s $(i < n)$ are in $Z_p[X]$, we can use them to define a sequence of divided powers, $1, \chi(P_{1,p}), \chi(P_{2,p}), \cdots, \chi(P_{n-1,p})$ in $k[X]$. By Theorem 2 and Sublemma 3 of [**1**], $1, \chi(P_{1,p}), \chi(P_{2,p}), \cdots, \chi(P_{n-1,p})$ has an extension in $k[X]$. Call the homogeneous component of degree n of that extension (which is also an extension) E. We will show that $\chi(P_{n,p}) = E$ (and in the process show that χ is well defined on $P_{n,p}$, i.e., the coefficients of $P_{n,p}$ are in Z_p).

Let $M \equiv \prod_{i=1}^{r} X_{p^i}^{t_i}$ be a monomial of degree n and index p^r. Assume its coefficient is c in $P_{n,p}$ and c' in E. We want to show that $\chi(c) = c'$. Assume that this statement is true for all monomials larger than M. (This set may be vacuous.) If M is of the form $X_{p^r}^{t_r}$, let $T \equiv X_{p^r}^{\alpha} \otimes X_{p^r}^{t_r-\alpha}$, where α is the integer found in Lemma 5 for $n = t_r$. If M is not of this form, let $T \equiv (\prod_{i=1}^{r-1} X_{p^i}^{t_i}) \otimes X_{p^r}^{t_r}$. Among the monomials of degree n, T will occur only in the diagonalization of M or terms of larger degree. (If T is of the first form, the statement is clear. Otherwise, use a slight modification of Lemma 1.) Remember $\Delta P_{n,p} = P_{n,p} \otimes 1 + 1 \otimes P_{n,p} + \sum_{i=1}^{n-1} P_{i,p} \otimes P_{n-i,p}$ and $\Delta E = E \otimes 1 + 1 \otimes E + \sum_{i=1}^{n-1} \chi(P_{i,p}) \otimes \chi(P_{n-i,p})$. Therefore, e, the coefficient of T in $\Delta P_{n,p}$ is in Z_p, and $\chi(e)$ is the coefficient of T in ΔE. Now let $\{M_i\}_{i \in I}$ be the set of monomials of degree n larger than M. Let the coefficient of M_i in $P_{n,p}$ be c_i, and let the coefficient of T in ΔM_i be b_i. Then, $e = cb + \sum_{i\in I} c_i b_i$, where we have let b be the coefficient of T in ΔM. It is clear that each $b_i \in Z$. Also by the second induction hypothesis, we know that each $c_i \in Z_p$ and that if c_i' is the coefficient of M_i in E, $\chi(c_i) = c_i'$. Therefore $cb \in Z_p$ and $\chi(cb) = c'b$. If $M = X_{p^r}^{t_r}$ and t_r is not a power of p, then $b = \binom{t_r}{\alpha}$ which we assumed not to be divisible by p. Therefore $b \in Z_p$, $cb \in Z_p \Rightarrow c \in Z_p$ and $\chi(c) = c'$. If $M = X_{p^r}^{t_r}$, t_r a power of p, then $cb \equiv 0 \pmod p$. But we assumed that $b = \binom{t_r}{\alpha}$ was not divisible by p^2. Therefore, the denominator of c is not a multiple of p, i.e., $c \in Z_p$ and $\chi(c) = c'$. And lastly, if M is not of the form $X_{p^r}^{t_r}$, $b = 1$ and we are done.

COROLLARY 8. *If x is a primitive in H, then a sequence of divided power can be constructed over x using the polynomials defined in Definition* 6(a). *The obstructions can only occur at powers of p.*

PROOF. Similar to Corollary 3, using Theorem 7 instead of Theorem 2.

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