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INTERPOLATION IN H^p SPACES[1]

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ABSTRACT. A construction is given to show that for each $p < \infty$ there is a sequence of points in the unit disk which fails to satisfy Carleson's well-known condition, but which admits an H^p interpolation to every bounded sequence.

A sequence $\{z_n\}$ of points in the open unit disk is said to be *uniformly separated* if

$$\prod_{n=1;n\neq k}^{\infty} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| \geqq \delta$$

for some $\delta > 0$. Carleson [1] showed that

$$\{\{f(z_n)\} : f \in H^\infty\} = l^\infty$$

if and only if $\{z_n\}$ is uniformly separated. Shapiro and Shields [4] then showed that, for $1 \leqq p < \infty$,

$$\{\{(1 - |z_n|)^{1/p} f(z_n)\} : f \in H^p\} = l^p$$

if and only if $\{z_n\}$ is uniformly separated, and Kabaila [3] extended this result to $0 < p < 1$. (See also [2, Chapter 9].)

Recently, A. K. Snyder [5] has proved the existence of a sequence $\{z_n\}$, not uniformly separated, such that

$$\{\{f(z_n)\} : f \in H^2\} \supset l^\infty.$$

In this note, we give a direct construction which is valid for any $p < \infty$.

THEOREM 1. *For each $p < \infty$, there is a sequence $\{z_n\}$ which is not uniformly separated, yet has the property that, for each $\{w_n\} \in l^\infty$, there exists $f \in H^p$ with $f(z_n) = w_n$, $n = 1, 2, \cdots$.*

The proof depends on the following simple lemma.

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LEMMA. *For each infinite Blaschke product* $B(z)$ *there is a sequence* $\{\zeta_n\}$ *such that* $|\zeta_n| < 1$, $|\zeta_n| \to 1$, $B(\zeta_n) \neq 0$, $B(\zeta_n) \to 0$, *and*

$$(1 - |\zeta_n|)^\alpha / B(\zeta_n) \to 0$$

for every $\alpha > 0$.

PROOF OF LEMMA. Let $\{a_n\}$ be the zeros of $B(z)$, arranged so that $|a_1| \leqq |a_2| \leqq \cdots$. Let $e^{i\theta}$ be a boundary point such that $|B(re^{i\theta})| \to 1$ as $r \to 1$. Join a_1 to $e^{i\theta}$ by a path Γ_1 composed of an arc of the circle $|z| = |a_1|$ and the radial segment from $|a_1|\, e^{i\theta}$ to $e^{i\theta}$. Then the function

$$F(z) = B(z) \log (1 - |z|)$$

is continuous in the open disk, vanishes at a_1, and tends to $-\infty$ as $z \to e^{i\theta}$ along Γ_1. Thus there is a point ζ_1 on Γ_1 where $F(\zeta_1) = -1$. Now choose a_n with $|a_n| > |\zeta_1|$, and join a_n to $e^{i\theta}$ by a path Γ_2 of the same type. Thus there exists ζ_2 on Γ_2 where $F(\zeta_2) = -1$. Continuing in this manner, we construct a sequence $\{\zeta_n\}$ with $|\zeta_n| \to 1$ and $F(\zeta_n) = -1$. This proves the lemma.

PROOF OF THEOREM. We may assume $p \geqq 1$. Let $\{a_n\}$ be an arbitrary uniformly separated sequence with $|a_1| \leqq |a_2| \leqq \cdots$, and let $B(z)$ be its associated Blaschke product. The lemma shows, after passing to a subsequence, that there is a sequence $\{\zeta_n\}$ with $\{|\zeta_n|\}$ increasing to 1, $B(\zeta_n) \to 0$, and

$$\sum_{n=1}^{\infty} (1 - |\zeta_n|)\, |B(\zeta_n)|^{-p} < \infty. \tag{1}$$

We may take $\{\zeta_n\}$ to be an exponential sequence, hence uniformly separated (see [2]). Let the sequences $\{a_n\}$ and $\{\zeta_n\}$ be combined to form a sequence $\{z_n\}$ with $|z_1| \leqq |z_2| \leqq \cdots$, so that $\{a_k\} = \{z_{n_k}\}$ and $\{\zeta_k\} = \{z_{m_k}\}$, say. Given $\{w_n\} \in l^\infty$, let $\{c_k\} = \{w_{n_k}\}$ and $\{\omega_k\} = \{w_{m_k}\}$. By Carleson's theorem, there exists $g \in H^\infty$ with $g(a_k) = c_k$, $k = 1, 2, \cdots$. If we can find $h \in H^p$ such that

$$g(\zeta_k) + B(\zeta_k) h(\zeta_k) = \omega_k, \qquad k = 1, 2, \cdots, \tag{2}$$

then the function $f = g + Bh$ will be the desired interpolating function. But (2) is equivalent to

$$(1 - |\zeta_k|)^{1/p} h(\zeta_k) = (1 - |\zeta_k|)^{1/p} [B(\zeta_k)]^{-1} [\omega_k - g(\zeta_k)]. \tag{3}$$

By (1), the right-hand side of (3) forms an l^p sequence, so by the theorem of Shapiro and Shields, there exists $h \in H^p$ for which (3), and therefore (2), holds. The sequence $\{z_n\}$ is not uniformly separated, since $B(\zeta_n) \to 0$.

The proof shows that one can adjoin suitable points to an *arbitrary* uniformly separated sequence to produce a sequence $\{z_n\}$ of the type described in the theorem. By a similar construction, one can prove the following theorem.

THEOREM 2. *For each pair* (p, q), $0 < q < p < \infty$, *there is a sequence* $\{z_n\}$, *not uniformly separated, such that for each* $\{w_n\} \in l^p$ *there exists* $f \in H^q$ *with*

$$(1 - |z_n|)^{1/p} f(z_n) = w_n, \qquad n = 1, 2, \cdots.$$

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