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A NOTE ON TWO POINT BOUNDARY PROBLEMS FOR NONLINEAR MATRIX DIFFERENTIAL SYSTEMS

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ABSTRACT. This note is concerned with second order, nonlinear matrix differential systems involving a parameter together with boundary conditions specified at two points. The objective is to establish sufficient conditions for the existence of eigenvalues for the system. The results presented here represent extensions of recent work of the author. The principal motivation is provided by W. M. Whyburn's work on nonlinear boundary problems for second order differential systems.

1. **Introduction.** This note is concerned with the system of $2n^2$ first-order, nonlinear differential equations:

$$y'_{ij} = \sum_{h=1}^{n} k_{ih}[x, y_{11}, y_{12}, \cdots, y_{nn}, z_{11}, z_{12}, \cdots, z_{nn}; \lambda] z_{hj},$$

$$z'_{ij} = \sum_{h=1}^{n} - g_{ih}[x, y_{11}, y_{12}, \cdots, y_{nn}, z_{11}, z_{12}, \cdots, z_{nn}; \lambda] y_{hj},$$

$1 \leqq i, j \leqq n$, defined on $X: a \leqq x \leqq b$, $L: \lambda_0 - \delta < \lambda < \lambda_0 + \delta$, $0 < \delta \leqq \infty$. Nonlinear systems of this form were introduced in [2] and have been represented in the matrix form

$$(1) \qquad Y' = K(x; Y; Z; \lambda)Z, \qquad Z' = -G(x; Y; Z; \lambda)Y.$$

Studies concerned with the oscillation of solutions of nonlinear matrix differential equations have been conducted by H. C. Howard [5], E. C. Tomastik [6] and the author [2], [3]. In addition to oscillation, [2] also contains some results establishing the existence of eigenvalues for certain simple two point boundary problems. The purpose of this note is to extend the results presented in [4], where the linear version of (1) is studied, to the nonlinear case, thereby improving the work in [2].

2. **The two point boundary problem.** We consider (1) together with the two point boundary conditions

$$(2) \qquad \begin{aligned} A(\lambda)Y(a, \lambda) - B(\lambda)Z(a, \lambda) &= 0, \\ \det [C(\lambda)Y(b, \lambda) - D(\lambda)Z(b, \lambda)] &= 0, \end{aligned}$$

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where A, B, C and D are $n \times n$ matrices of continuous real-valued functions on L. We assume, specifically, that the coefficients in the problem satisfy the following hypotheses:

H_1. The functions k_{ij} and g_{ij} are real-valued and satisfy conditions insuring the existence of a solution when appropriate initial conditions are specified.

H_2. Each of K and G is symmetric on XL for all pairs Y, Z.

H_3. K is positive definite on XL for all pairs Y, Z.

H_4. $A^*(\lambda)B(\lambda) \equiv B^*(\lambda)A(\lambda)$, $C^*(\lambda)D(\lambda) \equiv D^*(\lambda)C(\lambda)$ on L.

H_5. $\det [B(\lambda)] \neq 0$ and $\det[C(\lambda)B^*(\lambda) - D(\lambda)A^*(\lambda)] \neq 0$ on L.

H_6. $A^*(\lambda)A(\lambda) + B^*(\lambda)B(\lambda) \equiv C^*(\lambda)C(\lambda) + D^*(\lambda)D(\lambda) \equiv I$ on L.

(* denotes transpose and I the identity matrix.)

A solution pair $\{Y(x, \lambda), Z(x, \lambda)\}$ is called *nontrivial* if $Y^*Y + Z^*Z$ is positive definite on X for each λ on L. We seek to establish the existence of values of λ which correspond to a nontrivial solution pair of $\{Y(x, \lambda), Z(x, \lambda)\}$ of (1) satisfying (2). Such values of λ are called the *eigenvalues* of the system.

If $\{Y(x, \lambda), Z(x, \lambda)\}$ is a solution pair of (1) satisfying

$$Y(a, \lambda) \equiv B^*(\lambda), \qquad Z(a, \lambda) \equiv A^*(\lambda) \quad \text{on} \quad L, \tag{3}$$

then, as in the linear case (see e.g. [4]), hypotheses H_4 and H_6 imply that the pair is nontrivial and conjoined, i.e., $Y^*Z \equiv Z^*Y$ on X for each λ on L.

3. **Existence of eigenvalues.** In establishing the existence of eigenvalues for the system (1), (2), we make extensive use of the corresponding results which were derived for the linear case [4].

THEOREM 1. *Let* $\{Y(x, \lambda), Z(x, \lambda)\}$ *be a solution of* (1), (3). *Let* $\Omega(x, \lambda)$ *and* $q(\lambda)$ *be defined by*

$$\Omega(x, \lambda) = Z^*K(x; Y; Z; \lambda)Z + Y^*G(x, Y; Z; \lambda)Y, \tag{4}$$

and

$$q(x, \lambda) = 2\int_a^x \operatorname{tr} \Omega(t, \lambda)\, dt, \tag{5}$$

respectively. Then for each x *on* X, $q(x, \lambda) > -2n\pi$ *on* L.

PROOF. As noted above, a solution pair $\{Y(x, \lambda), Z(x, \lambda)\}$ of (1), (3) is nontrivial and conjoined. It now follows that the matrix $\Psi(x, \lambda)$ defined by

$$\Psi(x, \lambda) = (Z + iY)(Z - iY)^{-1} \tag{6}$$

exists and is unitary on X for each λ on L. Also, for each fixed λ, Ψ

satisfies the first order differential

$$\Psi' = 2i\Psi\Omega \tag{7}$$

where Ω is the Hermitian matrix given by (4). Let $\psi_j(x, \lambda), j = 1, 2, \cdots, n$, denote the characteristic roots of Ψ and $\omega_j(x, \lambda), j = 1, 2, \cdots, n$, their respective arguments, where it is assumed that $0 \leqq \omega_j(a, \lambda_0) < 2\pi, j = 1, 2, \cdots, n$, and that the functions $\omega_j(x, \lambda)$ are continued as continuous functions on XL. For each fixed λ, $\psi_j(x, \lambda) = +1$ for at least one j if and only if $\det Y(x, \lambda) = 0$, the number of $\psi_j(x, \lambda)$ having the value $+1$ equals the multiplicity of the zero of $\det Y(x, \lambda)$, the functions $\psi_j(x, \lambda)$ move positively on the unit circle at the point $+1$, and

$$q(x, \lambda) = 2\int_a^x \operatorname{tr} \Omega(t, \lambda)\, dt = \sum_{j=1}^n [\omega_j(x, \lambda) - \omega_j(a, \lambda)]. \tag{8}$$

The assumption H_5 implies $0 < \omega_j(a, \lambda) < 2\pi$ on L for $j = 1, 2, \cdots, n$. Since the characteristic roots $\omega_j(x, \lambda)$ move positively at the point $+1$ for each fixed λ, we have $\omega_j(x, \lambda) > 0$ on $X, j = 1, 2, \cdots, n$. Thus

$$q(x, \lambda) = \sum_{j=1}^n [\omega_j(x, \lambda) - \omega_j(a, \lambda)] > -\sum_{j=1}^n \omega_j(a, \lambda) > -2n\pi$$

on L for each x on X.

Our next theorem establishes a sufficient condition for the existence of eigenvalues for (1), (2).

THEOREM 2. *Let* $\{Y(x, \lambda), Z(x, \lambda)\}$ *be a solution of* (1), (3) *and let* $\Omega(x, \lambda)$ *and* $q(x, \lambda)$ *be defined by* (4) *and* (5). *Let* h *be an integer such that* g.l.b. $q(\lambda) \leqq 2h\pi$ *and let* m *be an integer such that* l.u.b. $q(\lambda) \geqq 2mn\pi$.

If $m > h + n$ *with* k *the largest nonnegative integer such that* $m - (h + n) \geqq kn$, *then there exist at least* k *nonempty sets of eigenvalues* $J_0, J_1, \cdots, J_{k-1}$ *for the system* (1), (2). *If* m *may be chosen arbitrarily large, then there exist infinitely many nonempty sets of eigenvalues* $J_0, J_1, \cdots$ *for the system* (1), (2). *In either case, the sets of eigenvalues* $J_0, J_1, \cdots$ *may be chosen so that if* $\lambda_p \in J_p$, $p \geqq 1$, *then the corresponding solution* $\{Y(x, \lambda_p), Z(x, \lambda_p)\}$ *has the property that* $\det Y(x, \lambda_p)$ *has at least* $p - 1$ *zeros on* X, *multiple zeros being counted according to their multiplicities.*

PROOF. Define the pair of matrices $\{U(x, \lambda), V(x, \lambda)\}$ on XL by

$$\begin{aligned} U(x, \lambda) &= C(\lambda)Y(x, \lambda) - D(\lambda)Z(x, \lambda), \\ V(x, \lambda) &= C(\lambda)Z(x, \lambda) + D(\lambda)Y(x, \lambda). \end{aligned} \tag{9}$$

It is readily verified using the properties of the pairs $\{Y, Z\}$ and $\{C, D\}$ that the pair $\{U, V\}$ has the properties: $U^*U + V^*V$ is positive definite and $U^*V = V^*U$ on X for each λ on L. The eigenvalues for the system

(1), (2) are the values of λ for which $\det U(b, \lambda) = 0$. As for solution pairs of (1), (3), it is readily verified that the matrix $\Phi(x, \lambda)$ defined by

$$\Phi(x, \lambda) = (V + iU)(V - iU)^{-1} \tag{10}$$

exists and is unitary on X for each λ on L. For each fixed λ on L, Φ satisfies the first order differential equation

$$\Phi' = 2i\Phi\bar{\Omega} \tag{11}$$

where $\bar{\Omega}$ is the Hermitian matrix

$$\bar{\Omega} = (C^* - iD^*)^{-1}\Omega(C + iD)^{-1} = (C + iD)\Omega(C + iD)^{-1}. \tag{12}$$

Letting $\varphi_j(x, \lambda)$, $j = 1, 2, \cdots, n$, denote the characteristic roots of Φ with $\beta_j(x, \lambda)$, $j = 1, 2, \cdots, n$, their respective arguments, where we again assume $0 \leqq \beta_j(a, \lambda_0) < 2\pi$ and each $\beta_j(x, \lambda)$ is continued as a continuous function on XL, we have, for each λ, $\varphi_j(x, \lambda) = +1$ for at least one j if and only if $\det U(x, \lambda) = 0$. We also have, since $\bar{\Omega}$ is similar to Ω,

$$\begin{aligned} \sum_{j=1}^{n} [\beta_j(x, \lambda) - \beta_j(a, \lambda)] &= 2\int_a^x \operatorname{tr} \bar{\Omega}(t, \lambda)\, dt = 2\int_a^x \operatorname{tr} \Omega(t, \lambda)\, dt \\ &= q(x, \lambda) = \sum_{j=1}^{n} [\omega_j(x, \lambda) - \omega_j(a, \lambda)] \end{aligned} \tag{13}$$

for each λ on L. In particular, in view of H_5, $0 < \beta_j(a, \lambda) < 2\pi$, $1 \leqq j \leqq n$, and so for $x = b$, we obtain, from (13),

$$q(b, \lambda) < \sum_{j=1}^{n} \beta_j(b, \lambda) < q(b, \lambda) + 2n\pi. \tag{14}$$

The remainder of the proof of the theorem can now be accomplished exactly as in the linear case. Specifically, let $\beta(\lambda) = \sum_{j=1}^{n} \beta_j(b, \lambda)$ and let h, k and m be the integers with the properties described in the hypothesis. Now, there exists a point $\bar{\lambda}$ on L such that $q(b, \bar{\lambda}) \leqq 2h\pi$ and there exists a point λ^* on L such that $q(b, \lambda^*) \geqq 2mn\pi$ with $m - (h + n) \geqq k \cdot n$. We assume $\bar{\lambda} < \lambda^*$. From (14) we have $\beta(\bar{\lambda}) \leqq 2(h + n)\pi$ and $\beta(\lambda^*) \geqq 2mn\pi$. By the continuity of $\beta(\lambda)$ on L, there exist $k + 1$ values of λ, $\lambda_0, \lambda_1, \cdots, \lambda_k$, with $\bar{\lambda} \leqq \lambda_0 < \lambda_1 < \cdots < \lambda_k \leqq \lambda^*$, such that $\beta(\lambda_p) = 2(h + n + pn)\pi$, $p = 0, 1, \cdots, k$. Now as λ increases from λ_p to λ_{p+1}, $0 \leqq p \leqq k - 1$, $\beta(\lambda)$ increases by $2n\pi$ and, consequently, at least one of the functions $\beta_j(b, \lambda)$ must increase by at least 2π so that $\beta_j(b, \lambda) \equiv 0 \pmod{2\pi}$ for some value of λ on the interval $[\lambda_p, \lambda_{p+1}]$. We conclude, therefore, that $\det U(b, \lambda) = 0$ for at least one value of λ on $[\lambda_p, \lambda_{p+1}]$ and the first part of the theorem follows. The remaining portions of the theorem are established exactly as in [**4**, Corollaries 2, 3].

We note that this theorem substantially improves the results in [**2**, §3] where very special two point boundary problems were considered and where it was required that each of the symmetric matrices K and G be positive definite on XL for all absolutely continuous pairs $\{Y, Z\}$. We note also that this theorem has a drawback not encountered in the linear case, namely, the conditions insuring the existence of eigenvalues depend upon the choice of a solution pair $\{Y, Z\}$ of (1), (3). To overcome this drawback we proceed as follows: Let Γ denote the collection to which the ordered pair $\{Y, Z\}$ belongs if and only if each of Y and Z is an $n \times n$ absolutely continuous matrix on XL, $Y^*Z \equiv Z^*Y$ and $Y^*Y + Z^*Z$ is positive definite on X for each λ on L. Define the $n \times n$ symmetric matrices S and T on $X\Gamma L$ by

$$\begin{aligned} S(x; Y; Z; \lambda) &= K(x; Y; Z; \lambda) + G(x; Y; Z; \lambda) \\ &\qquad - [K(x; Y; Z; \lambda) - G(X; Y; Z; \lambda)], \\ T(x; Y; Z; \lambda) &= K(x; Y; Z; \lambda) + G(x; Y; Z; \lambda) \\ &\qquad + [K(x; Y; Z; \lambda) - G(x; Y; Z; \lambda)], \end{aligned} \tag{15}$$

where $[M]$ denotes the nonnegative definite square root of M^*M. Extending the inequalities established by Atkinson [**1**, Chapter 10], we have

$$\operatorname{tr} S \leqq 2 \operatorname{tr} \Omega(x, \lambda) \leqq \operatorname{tr} T \tag{16}$$

whenever $\{Y, Z\}$ is a nontrivial conjoined solution of (1). For each λ on L, define $u(\lambda)$ and $v(\lambda)$ by

$$u(\lambda) = \text{g.l.b. } \operatorname{tr} S \text{ on } X\Gamma, \qquad v(\lambda) = \text{l.u.b. } \operatorname{tr} T \text{ on } X\Gamma. \tag{17}$$

THEOREM 3. *If there exist integers h and m with the properties*
(i) *$v(\bar{\lambda})(b - a) < 2h\pi$ for some $\bar{\lambda}$ on L,*
(ii) *$u(\lambda^*)(b - a) > 2mn\pi$ for some λ^* on L, and*
(iii) *$m > h + n$ with k the largest nonnegative integer such that*

$$m - (h + n) > kn,$$

then there exists at least k nonempty sets of eigenvalues $H_0, H_1, \cdots, H_{k-1}$ for the system (1), (2). *If the integer m may be chosen arbitrarily large, then there exist infinitely many nonempty sets of eigenvalues for the system.*

PROOF. Let $\{Y(x, \lambda), Z(x, \lambda)\}$ be any solution of (1), (3). Using the matrices defined in the proof of Theorem 1 and proceeding as in that proof, we have

$$\begin{aligned} v(\lambda)(b - a) &\geqq 2\int_a^b \operatorname{tr} \Omega(x, \lambda)\, dx = \sum_{j=1}^n [w_j(b, \lambda) - w_j(a, \lambda)] \\ &= \sum_{j=1}^n [\beta_j(b, \lambda) - \beta_j(a, \lambda)] > -2n\pi. \end{aligned}$$

Thus, for $\lambda = \bar{\lambda}$,

$$-2n\pi < 2\int_a^b \operatorname{tr} \Omega(x, \bar{\lambda})\, dx < 2h\pi.$$

Similarly, for $\lambda = \lambda^*$, $u(\lambda^*) > 2mn\pi$ implies $2 \int_a^b \operatorname{tr} \Omega(x, \lambda^*)\, dx > 2mn\pi$. Clearly, $\bar{\lambda} \neq \lambda^*$ since $\operatorname{tr} S \leqq \operatorname{tr} T$ on $X\Gamma L$. The proof is now completed exactly as in Theorem 2.

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