APPROXIMATION OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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Abstract. The results of this article are important for proving Riesz-type representation theorems for spaces of continuous functions with values in a topological vector space. It is well known that every continuous function with compact support from a locally compact Hausdorff space to a locally convex space can be uniformly approximated by continuous functions with finite-dimensional range. We give several conditions sufficient for this to be true without convexity. This problem is related to a vector-valued Tietze extension problem, and we give a new proof of a theorem of Dugundji, Arens, and Michael in this area, using topological tensor products.

Let $T$ be a compact Hausdorff space, $E$ a topological vector space (TVS) over either the real or complex field, and $C(T, E)$ the space of continuous functions from $T$ to $E$, with the topology of uniform convergence. When $E$ is the scalar field we write $C(T)$ instead of $C(T, E)$. For each $a \in C(T)$ and $x \in E$ the function $t \to a(t)x$ from $T$ to $E$, denoted by $a \otimes x$, is continuous. The linear span of these functions in $C(T, E)$ is the set of all finite sums $\sum a_i \otimes x_i$ with $a_i \in C(T)$ and $x_i \in E$ and is isomorphic to the algebraic tensor product $C(T) \otimes E$. If $E$ is locally convex, then $C(T) \otimes E$ is dense in $C(T, E)$ [4, Chapter III, §1, Proposition 1 and Lemma 2]. This property, the density property, is of prime importance in the representation of linear functionals and operators on $C(T, E)$ by vector measures [12]. The sufficient conditions given in this article have been used by the author [11] to extend these representations to the case when $E$ is not assumed to be locally convex. It is not known if any space $C(T, E)$ fails to have the density property.

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The results of this article extend to locally compact Hausdorff spaces. If $X$ is such a space, let $C(X, E)$ denote the space of continuous functions from $X$ to $E$ with compact support, with the topology of uniform convergence. Then $C(X) \otimes E$ is dense in $C(X, E)$ if $C(T, E)$ has the density property for every compact subspace $T \subset X$.

If $f = \sum a_i \otimes x_i \in C(T) \otimes E$, then $f$ has finite-dimensional range. When $E$ is a Hausdorff space this property characterizes $C(T) \otimes E$.

**Proposition 1.** If $E$ is a Hausdorff space, then $C(T) \otimes E$ consists of all functions in $C(T, E)$ having finite-dimensional range.

**Proof.** If $f \in C(T, E)$ and $\{x_1, \ldots, x_n\}$ is a basis for $f(T)$, then $f(T)$ is topologically isomorphic to the product of $n$ copies of the scalar field. The coefficient functionals $\lambda_i$ on $f(T)$ defined by $f(t) = \sum \lambda_i[f(t)]x_i$ are continuous, each $a_i = \lambda_i \cdot f$ is continuous and $f = \sum a_i \otimes x_i$. □

The following theorem is contained in a result of Turpin and Waelbroeck [14] on differentiable functions but our proof is much simpler for this special case. We generalize the usual proof of the density property in locally convex spaces to cover the case when $T$ has finite covering dimension [10]. If $\gamma$ is a covering of a topological space $X$, the order of $\gamma$ is

$$\sup_{x \in X} \sup \{n: x \text{ belongs to } n \text{ members of } \gamma\}.$$  

**Theorem 1.** If $T$ has finite covering dimension and $E$ is a TVS, then $C(T, E)$ has the density property.

**Proof.** Let $n$ be the dimension of $T$, $f \in C(T, E)$ and $U$ an open balanced 0-neighborhood in $E$. We will produce a function $g \in C(T) \otimes E$ such that $g - f$ maps $T$ into $U$. Each $t \in T$ lies in the open set $V_t = f^{-1}(f(t) + U)$ and a finite number of these sets, say $V_1, \ldots, V_m$, suffice to cover $T$.

Let $a_1, \ldots, a_m$ be a continuous partition of unity subordinate to this covering. Let $t_i \in V_i$ and $g = \sum_{i=1}^m a_i \otimes f(t_i)$. Then $g \in C(T) \otimes E$ and

$$g(t) - f(t) = \sum_{i=1}^m a_i(t)[f(t_i) - f(t)].$$  

If $t \in V_i$, then $f(t_i) - f(t) \in U$ and if $t \notin V_i$, then $a_i(t)[f(t_i) - f(t)] = 0$.

Since $U$ is balanced, each $g(t) - f(t)$ belongs to the $k$-fold sum $U + \cdots + U$, where $k$ is the number of sets $V_i$ containing $t$. Every open covering of $T$ has a refinement of order at most $n + 1$ that is also an open covering [10, pp. 9, 97] and since $T$ is compact we may assume that the refinement is finite. If we take such a refinement of the covering $\{V_i\}$ and define a partition of unity $(a_i)$ subordinate to this refinement, then
$g(t) - f(t)$ belongs to the $(n + 1)$-fold sum $U + \cdots + U$ for each $t \in T$. Since $n$ depends only on $T$, the proof is complete. □

We note that if $U$ is convex, then by (1), $g(t) - f(t) \in U$ for all $q$, so when $E$ is locally convex $C(T, E)$ has the density property regardless of the dimension of $T$.

Theorem 1 extends to compact spaces with an approximation-type property based on the uniform structure.

**Corollary 1.** If the identity map on $T$ can be uniformly approximated by continuous maps with finite-dimensional range, then $C(T, E)$ has the density property for each TVS $E$.

**Proof.** Let $(u_a)$ approximate the identity, $T_a$ be the quotient of $T$ by the relation \( \{(s, t) : u_a(s) = u_a(t)\} \), $\pi_a$ the quotient map from $T$ to $T_a$, $f \in C(T, E)$, and $f_a \in C(T_a, E)$ the map induced by $f \cdot u_a$. Since $T$ is compact, $T_a$ is homeomorphic to $u_a T$ and all the maps in question are uniformly continuous, so $(f \cdot u_a)$ converges uniformly to $f$. Also, each $f_a$ is the uniform limit of functions in $C(T_a) \otimes E$ and the maps from $T$ to $E$ induced by these functions belong to $C(T) \otimes E$ and converge uniformly to $f \cdot u_a$. □

**Corollary 2.** Let $T$ be imbedded in a cube $K$. If each $f \in C(T, E)$ has a continuous extension $f'$ to $K$, then $C(T, E)$ has the density property.

**Proof.** The identity map on a cube can be approximated uniformly by projections onto products of finitely many factors. Thus $f'$ can be approximated by mappings in $C(K) \otimes E$ and their restrictions to $T$ yield the desired result. □

Extensions of this type have been studied by several authors. In certain cases where $E$ is locally convex (and so $C(T, E)$ already has the density property), Arens [1, p. 15] and Dugundji [5, p. 37] have obtained extensions. For the general case, we make two definitions following Klee [7], [8]. Let $C$ be the class of all compact Hausdorff spaces. We say $E$ is admissible if $C(T, E)$ has the density property for all $T \in C$ (this is equivalent to Klee's definition) and is an extension space for $C$ if for all $K \in C$, every continuous map from a closed subset $T$ of $K$ into $E$ has a continuous extension to $K$. Then we have the following corollary.

**Corollary 3.** Every extension space for $C$ is admissible.

Recall that an $F$-space is a complete metrizable TVS. The next result is an immediate consequence of Corollary 3 and a theorem of Klee [7, p. 284].

**Corollary 4.** An $F$-space is admissible if and only if it is an extension space for $C$.  

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For locally convex spaces, topological tensor products and the density property can be used to prove "simultaneous" extension theorems for vector-valued functions directly from the corresponding theorems for real-valued functions. We illustrate this in the case of Corollary 2. The spaces involved are over the real field. The result is known [9, p. 802], but the present method simplifies the proof of the vector-valued case. If K is metrizable, then the map B below may be taken from Borsuk's theorem [3] and Arens' more complex result [1] need not be considered.

**Theorem 2.** Let T be a closed metric subspace of a compact Hausdorff space K. If E is a complete locally convex Hausdorff space, there is a topological isomorphism A of C(T, E) onto a subspace of C(K, E) that is an extension mapping, i.e., Af is an extension of f for every f ∈ C(T, E).

**Proof.** There is an isometric isomorphism B of C(T) onto a subspace of C(K) that is an extension mapping [1, p. 20]. Also, C(T, E) can be identified with C(T) ⊗ₚ E, the completion of the algebraic tensor product for the topology dₚ of bi-equicontinuous convergence, and C(S, E) with C(S) ⊗ₚ E [6, p. 90]. The map B ⊗ I, where I is the identity map on E, is a topological isomorphism and its extension A to C(T, E) answers our requirements [13, p. 440]. □

When E is not locally convex dₚ is no longer appropriate, but if E is a complete p-normed space, there is a topology for C(T) ⊗ E that is reminiscent of dₚ. In fact, C(T, E), as a complete p-normed space, is topologically isomorphic to a subspace of the p-normed space L(M(T), E). Here (see [11] for the integration theory) M(T) is the p-normed space of finite regular Borel measures on T that are of bounded variation as L(E, E)-valued measures, i.e., the p-norm

\[ \text{Var } \mu = \sup \sum |\mu(A_i)|^p, \]

taken over all finite partitions of T by Borel sets, is finite, and the isomorphism J is given by Jf(μ) = ∫ f dμ. However, we have not been able to show that the map B ⊗ I is an isomorphism.

Another class of conditions under which C(T, E) has the density property involves restrictions on E.

**Theorem 3.** Suppose the identity map on E can be approximated uniformly on precompact sets by continuous maps, where the range of each map has finite covering dimension. Then E is admissible.

**Proof.** If T ∈ C, f ∈ C(T, E) and U is a 0-neighborhood in E, there is a continuous map u on fT whose range S has finite covering dimension and ux = x ∈ U for all x ∈ fT. By Theorem 1, there is a map v ∈ C(S) ⊗ E such that vs = s ∈ U for all s ∈ S. Then g = v · u · f ∈ C(T) ⊗ E and g − f
maps $T$ into $U + U$. So $C(T, E)$ has the density property and $E$ is admissible. □

We define the approximation property for $E$ just as for locally convex spaces, i.e., the maps in Theorem 3 are linear and have range in a finite-dimensional linear subspace of $E$.

**Corollary 1.** If $E$ has the approximation property, then $E$ is admissible.

In an $F$-space, the "norm" $\|x\| = d(x, 0)$ defined by the metric $d$ is called an $F$-norm. We assume $d$ is translation-invariant and has balanced 0-spheres.

**Corollary 2** [8, p. 294]. If $E$ is an $F$-space with a basis, then $E$ is admissible.

**Proof.** By a theorem of Arsove [2] the coefficient functionals $F_n$ of a basis $(e_n)$ in $E$ are continuous, i.e., the basis is a Schauder basis. Applying the Banach-Steinhaus theorem to the maps $x \mapsto \sum F_n(x)e_n$, we see that $E$ has the approximation property. □

**Corollary 3.** $C(T, l^p)$ has the density property for all $p > 0$.

We extend this result to certain nonmetrizable spaces with bases. The topology of every TVS is determined by pseudo-metrics in the sense that their 0-spheres form a local subbase. It is clear that if $E$ has a determining family of complete metrics such that $E$ has a basis with respect to each one, then we can carry out the proof of Corollary 2 for each metric to show that $E$ is admissible. One way in which this situation arises is the following.

Let $E$ be a TVS with a basis $(e_n)$ and coefficient functionals $F_n$, which need not be continuous. Let $E$ have a determining family of complete metrics and $P$ be the corresponding family of $F$-norms. For each $p \in P$, let $E_p$ denote $E$ with the $p$-topology. Then $(e_n)$ need not be a basis for $E_p$, since the expansion of an element is unique in $E$ and converges in $E_p$ but may not be unique in $E_p$. In Theorem 4 we give a sufficient (and clearly necessary) condition for uniqueness.

**Theorem 4.** If for some $p \in P$, $x = \sum h_n e_n$ in $E_p$ implies

$$\sup_m p\left[\sum F_n(x)e_n\right] \leq \sup_m p\left[\sum h_n e_n\right],$$

then $(e_n)$ is a Schauder basis for $E_p$ and $E$. Thus if this is true for each $p \in P$, then $E$ is admissible.

**Proof.** We use techniques from [2]. The functional

$$p'(x) = \sup_m p\left[\sum F_n(x)e_n\right]$$

is well defined and determines a translation-invariant metric linear topology on $E$, $E_p'$, which denotes $E$ with the $p'$-topology, is stronger than $E_p$ since $p'$ dominates $p$. We show that in any case, $(e_n)$ is a Schauder basis for $E_p'$ with the same coefficient functionals as for $E$. For every $N$,

$$p'[x - \sum_{n=1}^{N} F_n(x)e_n] = \sup_p \left[ \sum_{n=N+1}^{\infty} F_n(x)e_n \right],$$

so that $\sum F_n(x)e_n$ converges to $x$ in $E_p'$. The representation is unique because if $\sum h_n e_n = 0$ in $E_p'$, then, for every $N$,

$$\lim_{m \to \infty} F_N(\sum_{n=1}^{m} h_n e_n) = 0.$$ 

But $F_N(\sum_{n=1}^{m} h_n e_n) = h_N$ for all $m \geq N$, so the coefficients are all zero. Since

$$p'[F_n(x)e_n] = p[F_n(x)e_n] \leq 2p'(x)$$

and $e_n \neq 0$, each $F_n$ is continuous on $E_p'$.

In order to complete the proof it suffices to show that $E_p = E_p'$, i.e., the identity map on $E$ is a homeomorphism for $p$ and $p'$. The set $S$ of scalar sequences $s = (h_n)$ such that $\sum h_n e_n$ converges in $E_p$ is a vector space and the sequences $x^* = (F_n(x))$ form a subspace $S'$. It is not difficult to show that $q(s) = \sup_m p[\sum_{n=1}^{m} h_n e_n]$ is an $F$-norm for an $F$-space topology on $S$ and that $x^* \leftrightarrow x$ is an isomorphism between $S'$ and $E_p'$. $S'$ and the subspace $S''$ of sequences $(h_n)$ for which $\sum h_n e_n = 0$ are algebraically complementary.

Let $s = (h_n) \in S$, $x = \sum h_n e_n$ and $[s]$ be the coset in $S/S''$ corresponding to $s$. The projection $s \mapsto x^*$ of $S$ onto $S'$ is continuous since, by hypothesis, $q(x^*) = p'(x) \leq q(s)$. Thus $S'$ and $S''$ are topological complements and $[s] \leftrightarrow x$ is an isomorphism between $S/S''$ and $E_p'$. The mapping $s \mapsto x$ from $S$ onto $E_p$ is linear and continuous, since $p(x) \leq p'(x) \leq q(s)$, and its kernel is $S''$. By the open mapping theorem, $[s] \leftrightarrow x$ is then an isomorphism between $S/S''$ and $E_p$ and the proof is complete. □

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