APPROXIMATION OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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Abstract. The results of this article are important for proving Riesz-type representation theorems for spaces of continuous functions with values in a topological vector space. It is well known that every continuous function with compact support from a locally compact Hausdorff space to a locally convex space can be uniformly approximated by continuous functions with finite-dimensional range. We give several conditions sufficient for this to be true without convexity. This problem is related to a vector-valued Tietze extension problem, and we give a new proof of a theorem of Dugundji, Arens, and Michael in this area, using topological tensor products.

Let $T$ be a compact Hausdorff space, $E$ a topological vector space (TVS) over either the real or complex field, and $C(T, E)$ the space of continuous functions from $T$ to $E$, with the topology of uniform convergence. When $E$ is the scalar field we write $C(T)$ instead of $C(T, E)$. For each $a \in C(T)$ and $x \in E$, the function $t \mapsto a(t)x$ from $T$ to $E$, denoted by $a \otimes x$, is continuous. The linear span of these functions in $C(T, E)$ is the set of all finite sums $\sum a_i \otimes x_i$ with $a_i \in C(T)$ and $x_i \in E$ and is isomorphic to the algebraic tensor product $C(T) \otimes E$. If $E$ is locally convex, then $C(T) \otimes E$ is dense in $C(T, E)$ [4, Chapter III, §1, Proposition 1 and Lemma 2]. This property, the density property, is of prime importance in the representation of linear functionals and operators on $C(T, E)$ by vector measures [12]. The sufficient conditions given in this article have been used by the author [11] to extend these representations to the case when $E$ is not assumed to be locally convex. It is not known if any space $C(T, E)$ fails to have the density property.

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The results of this article extend to locally compact Hausdorff spaces. If $X$ is such a space, let $C(X, E)$ denote the space of continuous functions from $X$ to $E$ with compact support, with the topology of uniform convergence. Then $C(X) \otimes E$ is dense in $C(X, E)$ if $C(T, E)$ has the density property for every compact subspace $T \subseteq X$.

If $f = \sum^n a_i \otimes x_i \in C(T) \otimes E$, then $f$ has finite-dimensional range. When $E$ is a Hausdorff space this property characterizes $C(T) \otimes E$.

**Proposition 1.** If $E$ is a Hausdorff space, then $C(T) \otimes E$ consists of all functions in $C(T, E)$ having finite-dimensional range.

**Proof.** If $f \in C(T, E)$ and $\{x_1, \ldots, x_n\}$ is a basis for $f(T)$, then $f(T)$ is topologically isomorphic to the product of $n$ copies of the scalar field. The coefficient functionals $\lambda_i$ on $f(T)$ defined by $f(t) = \sum \lambda_i[f(t)]x_i$ are continuous, each $a_i = \lambda_i \cdot f$ is continuous and $f = \sum a_i \otimes x_i$. □

The following theorem is contained in a result of Turpin and Waelbroeck [14] on differentiable functions but our proof is much simpler for this special case. We generalize the usual proof of the density property in locally convex spaces to cover the case when $T$ has finite covering dimension [10]. If $\gamma$ is a covering of a topological space $X$, the order of $\gamma$ is

$$\sup_{x \in X} \sup \{n : x \text{ belongs to } n \text{ members of } \gamma\}.$$ 

**Theorem 1.** If $T$ has finite covering dimension and $E$ is a TVS, then $C(T, E)$ has the density property.

**Proof.** Let $n$ be the dimension of $T$, $f \in C(T, E)$ and $U$ an open balanced 0-neighborhood in $E$. We will produce a function $g \in C(T) \otimes E$ such that $g - f$ maps $T$ into $U$. Each $t \in T$ lies in the open set $V_t = f^{-1}(f(t) + U)$ and a finite number of these sets, say $V_1, \ldots, V_m$, suffice to cover $T$.

Let $a_1, \ldots, a_m$ be a continuous partition of unity subordinate to this covering. Let $t_i \in V_i$ and $g = \sum_{i=1}^m a_i \otimes f(t_i)$. Then $g \in C(T) \otimes E$ and

$$g(t) - f(t) = \sum_{i=1}^m a_i(t)[f(t_i) - f(t)].$$

If $t \in V_i$, then $f(t_i) - f(t) \in U$ and if $t \notin V_i$, then $a_i(t)[f(t_i) - f(t)] = 0$.

Since $U$ is balanced, each $g(t) - f(t)$ belongs to the $k$-fold sum $U + \cdots + U$, where $k$ is the number of sets $V_i$ containing $t$. Every open covering of $T$ has a refinement of order at most $n + 1$ that is also an open covering [10, pp. 9, 97] and since $T$ is compact we may assume that the refinement is finite. If we take such a refinement of the covering $\{V_i\}$ and define a partition of unity $(a_i)$ subordinate to this refinement, then
$g(t) - f(t)$ belongs to the $(n + 1)$-fold sum $U + \cdots + U$ for each $t \in T$. Since $n$ depends only on $T$, the proof is complete. □

We note that if $U$ is convex, then by (1), $g(t) - f(t) \in U$ for all $q$, so when $E$ is locally convex $C(T, E)$ has the density property regardless of the dimension of $T$.

Theorem 1 extends to compact spaces with an approximation-type property based on the uniform structure.

**Corollary 1.** If the identity map on $T$ can be uniformly approximated by continuous maps with finite-dimensional range, then $C(T, E)$ has the density property for each TVS $E$.

**Proof.** Let $(u_n)$ approximate the identity, $T_\alpha$ be the quotient of $T$ by the relation $\{(s, t) : u_\alpha(s) = u_\alpha(t)\}$, $\pi_\alpha$ the quotient map from $T$ to $T_\alpha$, $f \in C(T, E)$, and $f_\alpha \in C(T_\alpha, E)$ the map induced by $f \cdot u_\alpha$. Since $T$ is compact, $T_\alpha$ is homeomorphic to $u_\alpha T$ and all the maps in question are uniformly continuous, so $(f \cdot u_\alpha)$ converges uniformly to $f$. Also, each $f_\alpha$ is the uniform limit of functions in $C(T_\alpha) \otimes E$ and the maps from $T$ to $E$ induced by these functions belong to $C(T) \otimes E$ and converge uniformly to $f \cdot u_\alpha$. □

**Corollary 2.** Let $T$ be imbedded in a cube $K$. If each $f \in C(T, E)$ has a continuous extension $f'$ to $K$, then $C(T, E)$ has the density property.

**Proof.** The identity map on a cube can be approximated uniformly by projections onto products of finitely many factors. Thus $f'$ can be approximated by mappings in $C(K) \otimes E$ and their restrictions to $T$ yield the desired result. □

Extensions of this type have been studied by several authors. In certain cases where $E$ is locally convex (and so $C(T, E)$ already has the density property), Arens [1, p. 15] and Dugundji [5, p. 37] have obtained extensions. For the general case, we make two definitions following Klee [7], [8]. Let $C$ be the class of all compact Hausdorff spaces. We say $E$ is admissible if $C(T, E)$ has the density property for all $T \in C$ (this is equivalent to Klee's definition) and is an extension space for $C$ if for all $K \in C$, every continuous map from a closed subset $T$ of $K$ into $E$ has a continuous extension to $K$. Then we have the following corollary.

**Corollary 3.** Every extension space for $C$ is admissible.

Recall that an $F$-space is a complete metrizable TVS. The next result is an immediate consequence of Corollary 3 and a theorem of Klee [7, p. 284].

**Corollary 4.** An $F$-space is admissible if and only if it is an extension space for $C$. 
For locally convex spaces, topological tensor products and the density property can be used to prove "simultaneous" extension theorems for vector-valued functions directly from the corresponding theorems for real-valued functions. We illustrate this in the case of Corollary 2. The spaces involved are over the real field. The result is known [9, p. 802], but the present method simplifies the proof of the vector-valued case. If $K$ is metrizable, then the map $B$ below may be taken from Borsuk's theorem [3] and Arens' more complex result [1] need not be considered.

**Theorem 2.** Let $T$ be a closed metric subspace of a compact Hausdorff space $K$. If $E$ is a complete locally convex Hausdorff space, there is a topological isomorphism $A$ of $C(T, E)$ onto a subspace of $C(K, E)$ that is an extension mapping, i.e., $Af$ is an extension of $f$ for every $f \in C(T, E)$.

**Proof.** There is an isometric isomorphism $B$ of $C(T)$ onto a subspace of $C(K)$ that is an extension mapping [1, p. 20]. Also, $C(T, E)$ can be identified with $C(T) \hat{\otimes} E$, the completion of the algebraic tensor product for the topology $\hat{\otimes}$ of bi-equicontinuous convergence, and $C(S, E)$ with $C(S) \hat{\otimes} E$ [6, p. 90]. The map $B \otimes I$, where $I$ is the identity map on $E$, is a topological isomorphism and its extension $A$ to $C(T, E)$ answers our requirements [13, p. 440]. □

When $E$ is not locally convex $\hat{\otimes}$ is no longer appropriate, but if $E$ is a complete $p$-normed space, there is a topology for $C(T) \otimes E$ that is reminiscent of $\hat{\otimes}$. In fact, $C(T, E)$, as a complete $p$-normed space, is topologically isomorphic to a subspace of the $p$-normed space $L(M(T), E)$. Here (see [11] for the integration theory) $M(T)$ is the $p$-normed space of finite regular Borel measures on $T$ that are of bounded variation as $L(E, E)$-valued measures, i.e., the $p$-norm

$$\text{Var } \mu = \sup \sum |\mu(A_i)|^p,$$

taken over all finite partitions of $T$ by Borel sets, is finite, and the isomorphism $J$ is given by $Jf(\mu) = \int f \, d\mu$. However, we have not been able to show that the map $B \otimes I$ is an isomorphism.

Another class of conditions under which $C(T, E)$ has the density property involves restrictions on $E$.

**Theorem 3.** Suppose the identity map on $E$ can be approximated uniformly on precompact sets by continuous maps, where the range of each map has finite covering dimension. Then $E$ is admissible.

**Proof.** If $T \in C, f \in C(T, E)$ and $U$ is a 0-neighborhood in $E$, there is a continuous map $u$ on $fT$ whose range $S$ has finite covering dimension and $ux = x \in U$ for all $x \in fT$. By Theorem 1, there is a map $v \in C(S) \otimes E$ such that $vs = s \in U$ for all $s \in S$. Then $g = v \cdot u \cdot f \in C(T) \otimes E$ and $g = f$
maps $T$ into $U + U$. So $C(T, E)$ has the density property and $E$ is admissible. □

We define the approximation property for $E$ just as for locally convex spaces, i.e., the maps in Theorem 3 are linear and have range in a finite-dimensional linear subspace of $E$.

**Corollary 1.** If $E$ has the approximation property, then $E$ is admissible. 

In an $F$-space, the "norm" $\|x\| = d(x, 0)$ defined by the metric $d$ is called an $F$-norm. We assume $d$ is translation-invariant and has balanced 0-spheres.

**Corollary 2** [8, p. 294]. If $E$ is an $F$-space with a basis, then $E$ is admissible.

**Proof.** By a theorem of Arsove [2] the coefficient functionals $F_n$ of a basis $(e_n)$ in $E$ are continuous, i.e., the basis is a Schauder basis. Applying the Banach-Steinhaus theorem to the maps $x \to \sum F_n(x)e_n$, we see that $E$ has the approximation property. □

**Corollary 3.** $C(T, l^p)$ has the density property for all $p > 0$.

We extend this result to certain nonmetrizable spaces with bases. The topology of every TVS is determined by pseudo-metrics in the sense that their 0-spheres form a local subbase. It is clear that if $E$ has a determining family of complete metrics such that $E$ has a basis with respect to each one, then we can carry out the proof of Corollary 2 for each metric to show that $E$ is admissible. One way in which this situation arises is the following.

Let $E$ be a TVS with a basis $(e_n)$ and coefficient functionals $F_n$, which need not be continuous. Let $E$ have a determining family of complete metrics and $P$ be the corresponding family of $F$-norms. For each $p \in P$, let $E_p$ denote $E$ with the $p$-topology. Then $(e_n)$ need not be a basis for $E_p$, since the expansion of an element is unique in $E$ and converges in $E_p$ but may not be unique in $E_p$. In Theorem 4 we give a sufficient (and clearly necessary) condition for uniqueness.

**Theorem 4.** If for some $p \in P$, $x = \sum h_n e_n$ in $E_p$ implies

$$\sup_m p \left[ \sum F_n(x)e_n \right] \leq \sup_m p \left[ \sum h_n e_n \right],$$

then $(e_n)$ is a Schauder basis for $E_p$ and $E$. Thus if this is true for each $p \in P$, then $E$ is admissible.

**Proof.** We use techniques from [2]. The functional

$$p'(x) = \sup_m p \left[ \sum F_n(x)e_n \right]$$
is well defined and determines a translation-invariant metric linear topology on \( E \). \( E_{p'} \), which denotes \( E \) with the \( p' \)-topology, is stronger than \( E_p \) since \( p' \) dominates \( p \). We show that in any case, \((e_n)\) is a Schauder basis for \( E_{p'} \) with the same coefficient functionals as for \( E \). For every \( N \),

\[
p' \left[ x - \sum_{n=1}^{N} F_n(x) e_n \right] = \sup_{m} \left[ \sum_{n=N+1}^{m} F_n(x) e_n \right],
\]

so that \( \sum F_n(x) e_n \) converges to \( x \) in \( E_{p'} \). The representation is unique because if \( \sum h_n e_n = 0 \) in \( E_{p'} \), then, for every \( N \),

\[
\lim_{m \to \infty} F_N \left( \sum_{n=1}^{m} h_n e_n \right) = 0.
\]

But \( F_N(\sum h_n e_n) = h_N \) for all \( m \geq N \), so the coefficients are all zero. Since

\[
p'[F_n(x) e_n] = p[F_n(x) e_n] \leq 2p'(x)
\]

and \( e_n \neq 0 \), each \( F_n \) is continuous on \( E_{p'} \).

In order to complete the proof it suffices to show that \( E_p = E_{p'} \), i.e., the identity map on \( E \) is a homeomorphism for \( p \) and \( p' \). The set \( S \) of scalar sequences \( s = (h_n) \) such that \( \sum h_n e_n \) converges in \( E_p \) is a vector space and the sequences \( x^* = (F_n(x)) \) form a subspace \( S' \). It is not difficult to show that \( q(s) = \sup_m p[\sum h_n e_n] \) is an \( F \)-norm for an \( F \)-space topology on \( S \) and that \( x^* \leftrightarrow x \) is an isomorphism between \( S' \) and \( E_{p'} \). \( S' \) and the subspace \( S'' \) of sequences \( (h_n) \) for which \( \sum h_n e_n = 0 \) are algebraically complementary.

Let \( s = (h_n) \in S \), \( x = \sum h_n e_n \) and \([s]\) be the coset in \( S/S'' \) corresponding to \( s \). The projection \( s \to x^* \) of \( S \) onto \( S' \) is continuous since, by hypothesis, \( q(x^*) = p'(x) \leq q(s) \). Thus \( S' \) and \( S'' \) are topological complements and \([s]\) is an isomorphism between \( S/S'' \) and \( E_{p'} \). The mapping \( s \to x \) from \( S \) onto \( E_p \) is linear and continuous, since \( p(x) \leq p'(x) \leq q(s) \), and its kernel is \( S'' \). By the open mapping theorem, \([s]\) is then an isomorphism between \( S/S'' \) and \( E_p \) and the proof is complete. \( \square \)

**References**


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