PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 1, January 1972

METRIC TRANSFORMS AND THE HYPERBOLIC FOUR-POINT PROPERTY

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ABSTRACT. The purpose of this paper is to show that any metric space is homeomorphic to a metric space with each quadruple of its points congruently imbeddable in three-dimensional hyperbolic space.

1. **Introduction.** Blumethal, Schoenberg, and others have made extensive investigations concerning metric transforms of euclidean and Hilbert spaces, as well as arbitrary metric spaces. The metric transform of a metric space may be defined as follows [**1**, p. 130].

DEFINITION 1.1. Let M be a metric space and $\phi(x)$ a real valued function defined for every value of $x = pq$, where p, q are points of M. A space $\phi(M)$ is the metric transform of M by ϕ provided (1) the points of M and $\phi(M)$ are in a one-to-one correspondence, and (2) if points p', q' of $\phi(M)$ correspond, respectively, to points p, q of M, then $p'q' = \phi(pq)$.

In this paper the space $\phi(M)$ has the same point-set as M, and the biuniform correspondence is the identity; i.e., $\phi(M)$ arises by redefining the distance pq of points p, q to be $\phi(pq)$.

Blumenthal [**2**, pp. 7–10] has shown that the metric transform $\phi(M)$ of any metric space M by $\phi(x) = x^\alpha$, $0 \leqq \alpha \leqq \frac{1}{2}$, has the euclidean four-point property.

In this paper we will show that the metric transform $\phi(M)$ of any metric space M by $\phi(x) = \cosh^{-1}(x^{2\alpha} + 1)$, $0 \leqq \alpha \leqq \frac{1}{2}$, has the hyperbolic four-point property.

In order to facilitate the arguments we introduce the following notation. The Cayley-Menger determinant of four points p_1, p_2, p_3, p_4 is defined by

$$D(p_1, p_2, p_3, p_4) = \begin{vmatrix} 0 & 1 \\ 1 & p_ip_j^2 \end{vmatrix} \qquad (i, j = 1, 2, 3, 4)$$

and the unbordered principal minor of order four of this determinant, $|p_ip_j^2|$ $(i, j = 1, 2, 3, 4)$ is denoted by $C(p_1, p_2, p_3, p_4)$. Since we will be

Received by the editors November 9, 1970.

AMS 1969 *subject classifications*. Primary 5250, 5040.

Key words and phrases. Cayley-Menger determinant, euclidean four-point property, hyperbolic four-point property, metric transform.

interested in the determinant obtained from $D(p_1, p_2, p_3, p_4)$, by adding the first row to the second, third, fourth, and fifth rows, respectively, and its unbordered principal minor of order four, we will use the notation

$$D'(p_1, p_2, p_3, p_4) = \begin{vmatrix} 0 & 1 \\ 1 & p_ip_j^2 + 1 \end{vmatrix}$$

and

$$C'(p_1, p_2, p_3, p_4) = |p_ip_j^2 + 1|.$$

We will denote the symmetric determinant $|\cosh p_ip_j|$ $(i, j = 1, 2, 3, 4)$ by $\Lambda(p_1, p_2, p_3, p_4)$.

2. **The metric transforms.** Let p_1, p_2, p_3 be any three points of a metric space. Since the function $\phi(x) = \cosh^{-1}(x^{2\alpha} + 1)$ is a monotone increasing concave function that vanishes at the origin, it is easily seen that $\phi(p_1p_2) + \phi(p_2p_3) > \phi(p_1p_3)$, for $0 \leqq \alpha \leqq \frac{1}{2}$. Thus, if p_1', p_2', p_3' are points with $p_i'p_j' = \cosh^{-1}[(p_ip_j)^{2\alpha} + 1]$ $(i, j = 1, 2, 3)$ where $0 \leqq \alpha \leqq \frac{1}{2}$ then p_1', p_2', p_3' are not collinear.

THEOREM 2.1. *The metric transform $\phi(M)$ of any metric space M by $\phi(x) = \cosh^{-1}(x^{2\alpha} + 1)$, $0 \leqq \alpha \leqq \frac{1}{2}$, has the hyperbolic four-point property.*

PROOF. First, suppose $\alpha = \frac{1}{2}$. Since M is metric so is $\phi(M)$ and it suffices to show that if p_1', p_2', p_3', p_4' are four points of $\phi(M)$, then $\Lambda(p_1', p_2', p_3', p_4') = |\cosh p_i'p_j'| = |p_ip_j + 1|$ $(i, j = 1, 2, 3, 4)$ is not positive, see [3, p. 224].

Now $\Lambda(\alpha) = |(p_ip_j)^{2\alpha} + 1|$ $(i, j = 1, 2, 3, 4)$ is a continuous function of α which is negative for $\alpha = 0$. If we suppose $\Lambda(\frac{1}{2}) > 0$, then a number α_0 exists $0 < \alpha_0 < \frac{1}{2}$, such that $\Lambda(\alpha_0) = 0$. It is known that

$$D(p_1'', p_2'', p_3'', p_4'') = \begin{vmatrix} 0 & 1 \\ 1 & (p_ip_j)^{2\alpha_0} \end{vmatrix} \qquad (i, j = 1, 2, 3, 4), 0 \leqq \alpha_0 < \tfrac{1}{2},$$

is positive [2, pp. 7–10], and hence $D'(p_1'', p_2'', p_3'', p_4'')$ is also positive. Denoting by [2, 1] the cofactor of the element in the second row and first column of $D'(p_1'', p_2'', p_3'', p_4'')$ a theorem of determinants gives

$$C'(p_1'', p_2'', p_3'', p_4'') \cdot D'(p_2'', p_3'', p_4'') - [2, 1]^2 \\ = D'(p_1'', p_2'', p_3'', p_4'') \cdot C'(p_2'', p_3'', p_4'').$$

But $C'(p_1'', p_2'', p_3'', p_4'') = \Lambda(p_1', p_2', p_3', p_4') = \Lambda(\alpha_0) = 0$, and $C'(p_2'', p_3'', p_4'')$ and $D'(p_1'', p_2'', p_3'', p_4'')$ are both positive and we are thus led to a contradiction. Therefore, $\Lambda(\frac{1}{2}) < 0$ and the theorem is proved for $\alpha = \frac{1}{2}$.

Since the above argument shows that $\Lambda(\alpha)$ cannot vanish for any value of α between zero and $\frac{1}{2}$, it follows that $\Lambda(\alpha) < 0$ for $0 < \alpha < \frac{1}{2}$. Hence the elements p_i' $(i = 1, 2, 3, 4)$ of $\phi(M)$ with

$$p_i'p_j' = \cosh^{-1}[(p_ip_j)^{2\alpha} + 1] \qquad (i, j = 1, 2, 3, 4),\ 0 < \alpha < \tfrac{1}{2},$$

are congruent with four points of three-dimensional hyperbolic space, and the theorem is proved.

We note that $\alpha = \frac{1}{2}$ is the greatest exponent for which the above theorem is valid. For example, let the points p_i $(i = 1, 2, 3, 4)$ form a pseudolinear quadruple with

$$p_1p_2 = p_2p_3 = p_3p_4 = p_1p_4 = 1, \qquad p_1p_3 = p_2p_4 = 2.$$

Now if this set is transformed by $\phi(x) = \cosh^{-1}[x^{2([1/2]+\varepsilon)} + 1]$ we find that

$$\Lambda(p_1', p_2', p_3', p_4') = [4 + 2 \cdot 2^{\varepsilon}][-2 \cdot 2^{2\varepsilon}][1 - (2 \cdot 2^{2\varepsilon} - 1)^2]$$

which vanishes for $\varepsilon = 0$ and is positive for $\varepsilon > 0$. Hence p_1', p_2', p_3', p_4' are congruent with four points of the hyperbolic plane for $\varepsilon = 0$, while if $\varepsilon > 0$, the four points are not congruently imbeddable in any hyperbolic space of curvature -1.

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