METRIC TRANSFORMS AND THE HYPERBOLIC FOUR-POINT PROPERTY

JOSEPH E. VALENTINE AND STANLEY G. WAYMENT

ABSTRACT. The purpose of this paper is to show that any metric space is homeomorphic to a metric space with each quadruple of its points congruently imbeddable in three-dimensional hyperbolic space.

1. Introduction. Blumenthal, Schoenberg, and others have made extensive investigations concerning metric transforms of euclidean and Hilbert spaces, as well as arbitrary metric spaces. The metric transform of a metric space may be defined as follows [1, p. 130].

DEFINITION 1.1. Let M be a metric space and \( \phi(x) \) a real valued function defined for every value of \( x = pq \), where \( p, q \) are points of \( M \). A space \( \phi(M) \) is the metric transform of \( M \) by \( \phi \) provided (1) the points of \( M \) and \( \phi(M) \) are in a one-to-one correspondence, and (2) if points \( p', q' \) of \( \phi(M) \) correspond, respectively, to points \( p, q \) of \( M \), then \( p'q' = \phi(pq) \).

In this paper the space \( \phi(M) \) has the same point-set as \( M \), and the biuniform correspondence is the identity; i.e., \( \phi(M) \) arises by redefining the distance \( pq \) of points \( p, q \) to be \( \phi(pq) \).

Blumenthal [2, pp. 7–10] has shown that the metric transform \( \phi(M) \) of any metric space \( M \) by \( \phi(x) = x^\alpha \), \( 0 \leq \alpha \leq \frac{1}{2} \), has the euclidean four-point property.

In this paper we will show that the metric transform \( \phi(M) \) of any metric space \( M \) by \( \phi(x) = \cosh^{-1}(x^{2\alpha} + 1) \), \( 0 \leq \alpha \leq \frac{1}{2} \), has the hyperbolic four-point property.

In order to facilitate the arguments we introduce the following notation.

The Cayley-Menger determinant of four points \( p_1, p_2, p_3, p_4 \) is defined by

\[
D(p_1, p_2, p_3, p_4) = \begin{vmatrix}
0 & 1 & 1 \\
1 & p_1p_2 & 1 \\
1 & 1 & p_3p_4 \\
1 & p_1p_3 & 1 \\
1 & 1 & p_2p_4 \\
1 & 1 & 1 \\
\end{vmatrix}
\]

and the unbordered principal minor of order four of this determinant, \( |p_ip_j| \) \( (i, j = 1, 2, 3, 4) \) is denoted by \( C(p_1, p_2, p_3, p_4) \). Since we will be
interested in the determinant obtained from $D(p_1, p_2, p_3, p_4)$, by adding the first row to the second, third, fourth, and fifth rows, respectively, and its unbordered principal minor of order four, we will use the notation

$$D'(p_1, p_2, p_3, p_4) = \begin{vmatrix} 0 & 1 \\ 1 & p_i p_j^2 + 1 \end{vmatrix}$$

and

$$C'(p_1, p_2, p_3, p_4) = |p_i p_j^2 + 1|.$$

We will denote the symmetric determinant $|\cosh p_i p_j|$ ($i, j = 1, 2, 3, 4$) by $\Lambda(p_1, p_2, p_3, p_4)$.

2. The metric transforms. Let $p_1, p_2, p_3$ be any three points of a metric space. Since the function $\phi(x) = \cosh^{-1}(x^{2x} + 1)$ is a monotone increasing concave function that vanishes at the origin, it is easily seen that $\phi(p_i, p_j) > \phi(p_k, p_l)$ for $0 \leq \alpha \leq \frac{1}{2}$. Thus, if $p_i', p_j', p_k', p_l'$ are points with $p_i' p_j' = \cosh^{-1}[(p_i p_j)^{2x} + 1]$ ($i, j = 1, 2, 3$) where $0 \leq \alpha \leq \frac{1}{2}$ then $p_i', p_j', p_k', p_l'$ are not collinear.

**Theorem 2.1.** The metric transform $\phi(M)$ of any metric space $M$ by $\phi(x) = \cosh^{-1}(x^{2x} + 1)$, $0 \leq \alpha \leq \frac{1}{2}$, has the hyperbolic four-point property.

**Proof.** First, suppose $\alpha = \frac{1}{2}$. Since $M$ is metric so is $\phi(M)$ and it suffices to show that if $p_i', p_j', p_k', p_l'$ are four points of $\phi(M)$, then $\Lambda(p_i', p_j', p_k', p_l') = |\cosh(p_i' p_j')| = |p_i p_j + 1|$ ($i, j = 1, 2, 3, 4$) is not positive, see [3, p. 224].

Now $\Lambda(\alpha) = |(p_i p_j)^{2x} + 1|$ ($i, j = 1, 2, 3, 4$) is a continuous function of $\alpha$ which is negative for $\alpha = 0$. If we suppose $\Lambda(\alpha) > 0$, then a number $\alpha_0$ exists $0 < \alpha_0 < \frac{1}{2}$, such that $\Lambda(\alpha_0) = 0$. It is known that

$$D(p_i', p_j', p_k', p_l') = \begin{vmatrix} 0 & 1 \\ 1 & (p_i p_j)^{2x_0} \end{vmatrix}$$

is positive [2, pp. 7-10], and hence $D'(p_i', p_j', p_k', p_l')$ is also positive. Denoting by [2, 1] the cofactor of the element in the second row and first column of $D'(p_i', p_j', p_k', p_l')$ a theorem of determinants gives

$$C'(p_i', p_j', p_k', p_l') \cdot D'(p_i', p_j', p_k', p_l') = [2, 1]^2$$

But $C'(p_i', p_j', p_k', p_l') = \Lambda(p_i', p_j', p_k', p_l') = \Lambda(\alpha_0) = 0$, and $C'(p_i', p_j', p_k', p_l')$ and $D'(p_i', p_j', p_k', p_l')$ are both positive and we are thus led to a contradiction. Therefore, $\Lambda(\alpha) < 0$ and the theorem is proved for $\alpha = \frac{1}{2}$.

Since the above argument shows that $\Lambda(\alpha)$ cannot vanish for any value of $\alpha$ between zero and $\frac{1}{2}$, it follows that $\Lambda(\alpha) < 0$ for $0 < \alpha < \frac{1}{2}$. Hence
the elements $p_i'$ ($i = 1, 2, 3, 4$) of $\phi(M)$ with
\[ p_i' p_j' = \cosh^{-1} [(p_i p_j)^{2\alpha} + 1] \quad (i, j = 1, 2, 3, 4), \quad 0 < \alpha < \frac{1}{2}, \]
are congruent with four points of three-dimensional hyperbolic space, and the theorem is proved.

We note that $\alpha = \frac{1}{2}$ is the greatest exponent for which the above theorem is valid. For example, let the points $p_i$ ($i = 1, 2, 3, 4$) form a pseudolinear quadruple with
\[ p_1 p_2 = p_3 p_4 = p_1 p_4 = 1, \quad p_1 p_3 = p_2 p_4 = 2. \]
Now if this set is transformed by $\phi(x) = \cosh^{-1} \left[ x^{2(1/2)+\epsilon} + 1 \right]$ we find that
\[ \Lambda(p_1', p_2', p_3', p_4') = [4 + 2 \cdot 2^\epsilon][-2 \cdot 2^2\epsilon][1 - (2 \cdot 2^2\epsilon - 1)^2] \]
which vanishes for $\epsilon = 0$ and is positive for $\epsilon > 0$. Hence $p_1', p_2', p_3', p_4'$ are congruent with four points of the hyperbolic plane for $\epsilon = 0$, while if $\epsilon > 0$, the four points are not congruently imbeddable in any hyperbolic space of curvature $-1$.

REFERENCES


DEPARTMENT OF MATHEMATICS, UTAH STATE UNIVERSITY, LOGAN, UTAH 84321