SOME EXAMPLES RELATING THE DELETED PRODUCT TO IMBEDDABILITY

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Abstract. Examples are given showing the limitations of the homology of the deleted product in determining the imbeddability of simplicial complexes in a given Euclidean space. It is also proven that the only finite 1-complexes whose polyhedral deleted products are closed 2-manifolds are the two primitive skew curves of Kuratowski.

1. Introduction. The deleted product of a topological space $X$ is defined to be the space

$$X^* = \{(x_1, x_2) \in X \times X \mid x_1 \neq x_2\}.$$ 

In [7] we modified the techniques developed by C. W. Patty in [5] to prove the following theorem which is a version of Theorem 2 of [6]:

**Theorem 1.1.** A collapsible 2-dimensional simplicial complex $X$ can be imbedded in Euclidean 2-space if and only if $H_2(X^*) = 0$ and $H_3(X^*) = 0$, where $H_\ast(\cdot)$ denotes singular homology with integer coefficients.

In this article we present examples which show that the homology of the deleted product fails to distinguish imbeddability if either a larger class of spaces $X$ is considered or if imbeddability in 2-space is weakened to imbeddibility in 3-space. Specifically, we prove:

**Proposition 1.2.** There are finite 1-dimensional simplicial complexes $A_1$ and $A_2$ such that $H_\ast(A_1^*) \cong H_\ast(A_2^*)$, $H_\ast(A_1) \cong H_\ast(A_2)$, $A_1$ is planar (i.e., imbeddable in $R^2$) but $A_2$ is nonplanar.

**Proposition 1.3.** There are collapsible 2-dimensional simplicial complexes $B_1$ and $B_2$ such that $H_\ast(B_1^*) \cong H_\ast(B_2^*)$, $B_1$ can be imbedded in $R^3$, but $B_2$ cannot be imbedded in $R^3$.

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We also show that Theorem 1.1 is the best possible result in the following sense:

**Proposition 1.4.** There are nonplanar collapsible 2-dimensional simplicial complexes \( C_1 \) and \( C_2 \) such that \( H_2(C_1^*) \neq 0 \), \( H_3(C_1^*) = 0 \), \( H_2(C_2^*) = 0 \), and \( H_3(C_2) \neq 0 \).

The complexes \( A_2, B_2, \) and \( C_2 \) are constructed from the 1-dimensional complexes \( K_5 \) (the complete graph on five vertices) and \( K_{3,3} \) (the complete bipartite graph on two sets of three vertices). It is proven in [2] that \( P(K_5^*) \) is a 2-sphere with six handles while \( P(K_{3,3}^*) \) is a 2-sphere with four handles, where \( P(X^*) = \{(x_1, x_2) \in X^* | x_1 \) and \( x_2 \) lie in disjoint closed simplices of \( X \} \) is the polyhedral deleted product of the finite complex \( X \) and is a deformation retract of \( X^* \) (cf. [3]). We conclude the paper with the following theorem which parallels Kuratowski's theorem (that a finite 1-complex is nonplanar if and only if it has a subcomplex homeomorphic to either \( K_5 \) or \( K_{3,3} \)):

**Theorem 1.5.** If \( X \) is a finite 1-dimensional simplicial complex, then \( P(X^*) \) is a closed 2-manifold if and only if \( X \) is \( K_5 \) or \( K_{3,3} \).

2. **Computational lemmas.** In [4], C. W. Patty attempted to provide a stepwise algorithm for computing the homology of the deleted product of an arbitrary finite 1-dimensional complex. Unfortunately, two crucial parts (Theorems 4.1 and 4.2) of his algorithm are incorrect, as can be seen by considering the complex \( K_5 \); using Patty's algorithm one would conclude that the second homology of \( K_5^* \) has even rank and the first homology of \( K_5^* \) has odd rank. But since \( P(K_5^*) \) is a 2-sphere with six handles, \( H_2(K_5^*) \) has rank one and \( H_1(K_5^*) \) has rank twelve.

For our purposes, it suffices to list three results which provide an algorithm for computing the homology of the deleted product of certain finite 1-complexes. The first is a restatement of Theorems 3.1 and 3.2 of [4], the second is a weakened (but valid) version of Theorems 4.1 and 4.2 of [4], and the third is a simple piecing together lemma. For proofs see [7]. Let \( Z^k \) denote the direct sum of \( k \) copies of the integers. The degree of a vertex \( v \) of a 1-complex \( X \) is the number of 1-simplices of \( X \) having \( v \) as a vertex.

**Proposition 2.1.** If \( X \) is the simplicial complex which is constructed by adding a 1-simplex to the 1-complex \( Y \) at a vertex of degree \( n \) in \( Y \), then if \( Y \) is not an arc
\[
\begin{align*}
H_1(X^*) &\cong H_1(Y^*) \oplus Z^{2n-2}, \\
H_j(X^*) &\cong H_j(Y^*) \quad \text{if } j \neq 1.
\end{align*}
\]
Proposition 2.2. Suppose the simplicial complex $X$ is constructed by adding a 1-simplex between the vertices $v_1$ and $v_2$ of the connected 1-complex $Y$, $Y$ is not an arc. If every simple closed curve in $W = X - \bigcup_{i=1}^2 st(v_i, X)$ can be written as the sum (in the homology sense) of simple closed curves in $W$ which do not separate $v_1$ and $v_2$ in $Y$, then

$$H_i(X^*) \cong H_i(Y^*) \oplus H_0(W) \oplus H_0(W),$$

$$H_2(X^*) \cong H_2(Y^*) \oplus H_1(W) \oplus H_1(W),$$

$$H_3(X^*) \cong H_3(Y^*) \quad \text{if} \quad j \neq 1, 2.$$

Proposition 2.3. Suppose the 1-complex $X$ is the union of two connected subcomplexes $X_1$ and $X_2$ such that $X_1 \cap X_2$ is a 1-simplex $S$ whose midpoint separates $X_1 - S$ and $X_2 - S$ in $X$. If neither $X_1$ nor $X_2$ is an arc, then

$$H_i(X^*) \cong H_i(X_1^*) \oplus H_i(X_2^*) \oplus \mathbb{Z},$$

$$H_2(X^*) \cong H_2(X_1^*) \oplus H_2(X_2^*) \oplus H_2(X_1 \times X_2) \oplus H_2(X_1 \times X_2),$$

$$\tilde{H}_j(X^*) = 0 \quad \text{if} \quad j \neq 1, 2.$$

The 2-dimensional complexes of our examples are cones over 1-complexes, and we will need the following homology version of a theorem of A. H. Copeland. Jr. (cf. [1]). Let $CX$ denote the cone over $X$, which is taken to be the quotient space of $X \times I$ under the identification of $(x_1, 1)$ with $(x_2, 1)$ for all $x_1$ and $x_2$ in $X$. The equivalence class of $CX$ containing $(x, t)$ is denoted by $[x, t]$.

Proposition 2.4. If $X$ is a finite simplicial complex, then there is an exact sequence

$$\cdots \longrightarrow H_i(X^*) \xrightarrow{\phi_i} H_i(X) \oplus H_i(X) \xrightarrow{\psi_i} H_i((CX)^*) \xrightarrow{\Delta} H_{i-1}(X^*) \longrightarrow \cdots$$

in which

$$\phi_i(u) = (p_1(u), p_2(u)), \quad u \in H_i(X^*),$$

$$\psi_i(u_1, u_2) = g_{1*}(u_1) = g_{2*}(u_2), \quad u_1, u_2 \in H_i(X),$$

where $p_i : X^* \to X$ is defined by $p_i(x_1, x_2) = x_i$ and $g_i : X \to (CX)^*$ is defined by $g_i(x) = ([x, 2 - i], [x, i - 1])$.

3. The examples. The 1-dimensional simplicial complexes $A_1$ and $A_2$ of Proposition 1.2 are drawn in Figure 3.1. $H_*(A_1^*)$ is computed by starting with a simple closed curve and adding 1-simplices one at a time so that at each stage either Proposition 2.1 or 2.2 applies. Since $A_2$ consists of two copies of $K_{3,3}$ joined by a 1-simplex, $H_*(A_2^*)$ is easily computed using the remarks of §1 and Propositions 2.1
and 2.3. The result of these calculations is:

\[ \tilde{H}_j(A_1^*') \cong \tilde{H}_j(A_2^*') = 0 \quad \text{for } j \neq 1, 2, \]
\[ H_1(A_1^*) \cong H_1(A_2^*) \cong \mathbb{Z}^{25}, \]
\[ H_2(A_1^*) \cong H_2(A_2^*) \cong \mathbb{Z}^{34}, \]

we also have

\[ \tilde{H}_j(A_1') \cong \tilde{H}_j(A_2') = 0 \quad \text{for } j \neq 1, \quad H_1(A_1') \cong H_1(A_2') \cong \mathbb{Z}^8. \]

Since \( A_1 \) is planar while \( A_2 \) is nonplanar, this proves Proposition 1.2.

Now set \( B_1 = CA_1 \) and \( B_2 = CA_2 \). Then \( B_1 \) and \( B_2 \) are collapsible 2-dimensional simplicial complexes, \( B_1 \) is imbeddable in \( \mathbb{R}^3 \) while \( B_2 \) is not. Using Proposition 2.4 we have

\[ \tilde{H}_j(B_1^*') \cong \tilde{H}_j(B_2^*') = 0 \quad \text{for } j \neq 2, 3, \]
\[ H_2(B_1^*) \cong H_2(B_2^*) \cong \mathbb{Z}^9, \]
\[ H_3(B_1^*) \cong H_3(B_2^*) \cong \mathbb{Z}^{24} \]

This proves Proposition 1.3.

Finally set \( C_1 = CF \) where \( F \) is the disjoint union of a point and a simple closed curve, and let \( C_2 = CK_5 \). Then using Proposition 2.4 we have

\[ H_2(C_1^*) \cong \mathbb{Z}, \quad H_2(C_2^*) = 0, \]
\[ H_3(C_1^*) = 0, \quad H_3(C_2^*) \cong \mathbb{Z}. \]

Since both \( C_1 \) and \( C_2 \) are collapsible 2-dimensional nonplanar simplicial complexes, this proves Proposition 1.4.

![Figure 3.1. The Complexes A1 (top) and A2.](image-url)
4. **Proof of Theorem 1.5.** Suppose \( X \) is a finite 1-dimensional simplicial complex such that \( P(X^*) \) is a closed 2-manifold. We will show that either \( X = K_5 \) or \( X = K_{3,3} \).

First observe that \( X \) must be connected. For if \( A \) and \( B \) were distinct components of \( X \), then \( A \times B \) and \( P(A^*) \) would be components of \( P(X^*) \) and hence closed 2-manifolds. \( A \times B \) could be a closed 2-manifold only if \( A \) is a simple close curve, but then \( P(A^*) \) would not be a 2-manifold.

Since every 1-cell of \( P(X^*) \) must be a face of exactly two 2-cells of \( P(X^*) \) we can conclude:

(i) \( X \) contains no vertices of degree less than three or greater than four.

(ii) The closed star of a vertex of degree four in \( X \) contains both vertices of every 1-simplex of \( X \).

(iii) The closed star of a vertex of degree three in \( X \) contains a vertex of every one simplex of \( X \).

If \( X \) contains a vertex of degree four, then (ii) implies that \( X \) has exactly five vertices. So \( X \subseteq K_5 \) and hence \( P(X^*) \subseteq P(K_5^*) \). Since \( P(X^*) \) and \( P(K_5^*) \) are closed manifolds we must have \( P(X^*) = P(K_5^*) \) and hence \( X = K_5 \).

Finally, suppose every vertex of \( X \) has degree three. By the preceding case, \( X \) has more than five vertices. Choose a vertex \( u_0 \) of \( X \) and let \( u_1, u_2, \) and \( u_3 \) be the other three vertices of the closed star of \( u_0 \) in \( X \). If \( w_1 \) and \( w_2 \) are two other vertices of \( X \), then (iii) implies that each 1-simplex meeting \( w_i \) must also meet some \( u_j, j > 0 \). This accounts for three 1-simplices meeting each of these six vertices, so \( X \) contains no other 1-simplices or vertices. The 1-complex we have constructed is exactly \( K_{3,3} \).

**References**


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