SOME EXAMPLES RELATING THE DELETED PRODUCT TO IMBEDDABILITY

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Abstract. Examples are given showing the limitations of the homology of the deleted product in determining the imbeddability of simplicial complexes in a given Euclidean space. It is also proven that the only finite 1-complexes whose polyhedral deleted products are closed 2-manifolds are the two primitive skew curves of Kuratowski.

1. Introduction. The deleted product of a topological space $X$ is defined to be the space

$$X^* = \{(x_1, x_2) \in X \times X \mid x_1 \neq x_2\}.$$

In [7] we modified the techniques developed by C. W. Patty in [5] to prove the following theorem which is a version of Theorem 2 of [6]:

**Theorem 1.1.** A collapsible 2-dimensional simplicial complex $X$ can be imbedded in Euclidean 2-space if and only if $H_2(X^*) = 0$ and $H_3(X^*) = 0$, where $H_\ast(\_)$ denotes singular homology with integer coefficients.

In this article we present examples which show that the homology of the deleted product fails to distinguish imbeddability if either a larger class of spaces $X$ is considered or if imbeddability in 2-space is weakened to imbeddibility in 3-space. Specifically, we prove:

**Proposition 1.2.** There are finite 1-dimensional simplicial complexes $A_1$ and $A_2$ such that $H_\ast(A_1^*) \cong H_\ast(A_2^*)$, $H_\ast(A_1) \cong H_\ast(A_2)$, $A_1$ is planar (i.e., imbeddable in $R^2$) but $A_2$ is nonplanar.

**Proposition 1.3.** There are collapsible 2-dimensional simplicial complexes $B_1$ and $B_2$ such that $H_\ast(B_1^*) \cong H_\ast(B_2^*)$, $B_1$ can be imbedded in $R^3$, but $B_2$ cannot be imbedded in $R^3$.

Received by the editors November 4, 1970.


Key words and phrases. Deleted product, imbedding in Euclidean space, simplicial complex.

1 This research was partially supported by the National Science Foundation and is extracted from the author's Ph.D. thesis at the University of Wisconsin under Professor Edward Fadell.
We also show that Theorem 1.1 is the best possible result in the following sense:

**Proposition 1.4.** There are nonplanar collapsible 2-dimensional simplicial complexes $C_1$ and $C_2$ such that $H_2(C_1^*) \neq 0$, $H_3(C_1^*) = 0$, $H_2(C_2^*) = 0$, and $H_3(C_2) \neq 0$.

The complexes $A_2$, $B_2$, and $C_2$ are constructed from the 1-dimensional complexes $K_5$ (the complete graph on five vertices) and $K_{3,3}$ (the complete bipartite graph on two sets of three vertices). It is proven in [2] that $P(K_5^*)$ is a 2-sphere with six handles while $P(K_{3,3}^*)$ is a 2-sphere with four handles, where $P(X^*) = \{(x_1, x_2) \in X^* | x_1$ and $x_2$ lie in disjoint closed simplices of $X\}$ is the polyhedral deleted product of the finite complex $X$ and is a deformation retract of $X^*$ (cf. [3]). We conclude the paper with the following theorem which parallels Kuratowski's theorem (that a finite 1-complex is nonplanar if and only if it has a subcomplex homeomorphic to either $K_5$ or $K_{3,3}$):

**Theorem 1.5.** If $X$ is a finite 1-dimensional simplicial complex, then $P(X^*)$ is a closed 2-manifold if and only if $X$ is $K_5$ or $K_{3,3}$.

2. Computational lemmas. In [4], C. W. Patty attempted to provide a stepwise algorithm for computing the homology of the deleted product of an arbitrary finite 1-dimensional complex. Unfortunately, two crucial parts (Theorems 4.1 and 4.2) of his algorithm are incorrect, as can be seen by considering the complex $K_5$; using Patty's algorithm one would conclude that the second homology of $K_5^*$ has even rank and the first homology of $K_5^*$ has odd rank. But since $P(K_5^*)$ is a 2-sphere with six handles, $H_2(K_5^*)$ has rank one and $H_1(K_5^*)$ has rank twelve.

For our purposes, it suffices to list three results which provide an algorithm for computing the homology of the deleted product of certain finite 1-complexes. The first is a restatement of Theorems 3.1 and 3.2 of [4], the second is a weakened (but valid) version of Theorems 4.1 and (4.2) of [4], and the third is a simple piecing together lemma. For proofs see [7]. Let $Z^k$ denote the direct sum of $k$ copies of the integers. The degree of a vertex $v$ of a 1-complex $X$ is the number of 1-simplices of $X$ having $v$ as a vertex.

**Proposition 2.1.** If $X$ is the simplicial complex which is constructed by adding a 1-simplex to the 1-complex $Y$ at a vertex of degree $n$ in $Y$, then if $Y$ is not an arc

\[ H_1(X^*) \cong H_1(Y^*) \oplus Z^{2n-2}, \]

\[ H_j(X^*) \cong H_j(Y^*) \text{ if } j \neq 1. \]
Proposition 2.2. Suppose the simplicial complex $X$ is constructed by adding a 1-simplex between the vertices $v_1$ and $v_2$ of the connected 1-complex $Y$. $Y$ is not an arc. If every simple closed curve in $W = X - \bigcup_{i=1}^{2} st(v_i, X)$ can be written as the sum (in the homology sense) of simple closed curves in $W$ which do not separate $v_1$ and $v_2$ in $Y$, then
\[
\begin{align*}
H_1(X^*) &\cong H_1(Y^*) \oplus H_0(W) \oplus H_0(W), \\
H_2(X^*) &\cong H_2(Y^*) \oplus H_1(W) \oplus H_1(W), \\
H_j(X^*) &\cong H_j(Y^*) \quad \text{if } j \neq 1, 2.
\end{align*}
\]

Proposition 2.3. Suppose the 1-complex $X$ is the union of two connected subcomplexes $X_1$ and $X_2$ such that $X_1 \cap X_2$ is a 1-simplex $S$ whose midpoint separates $X_1 - S$ and $X_2 - S$ in $X$. If neither $X_1$ nor $X_2$ is an arc, then
\[
\begin{align*}
H_1(X^*) &\cong H_1(X_1^*) \oplus H_1(X_2^*) \oplus \mathbb{Z}, \\
H_2(X^*) &\cong H_2(X_1^*) \oplus H_2(X_2^*) \oplus H_2(X_1 \times X_2) \oplus H_2(X_1 \times X_2), \\
H_j(X^*) &\cong 0 \quad \text{if } j \neq 1, 2.
\end{align*}
\]

The 2-dimensional complexes of our examples are cones over 1-complexes, and we will need the following homology version of a theorem of A. H. Copeland, Jr. (cf. [1]). Let $CX$ denote the cone over $X$, which is taken to be the quotient space of $X \times I$ under the identification of $(x_1, 1)$ with $(x_2, 1)$ for all $x_1$ and $x_2$ in $X$. The equivalence class of $CX$ containing $(x, t)$ is denoted by $[x, t]$.

Proposition 2.4. If $X$ is a finite simplicial complex, then there is an exact sequence
\[
\cdots \longrightarrow H_i(X^*) \xrightarrow{\phi_i} H_i(X) \oplus H_i(X) \xrightarrow{\psi_i} H_i((CX)^*) \xrightarrow{\Delta} H_{i-1}(X^*) \longrightarrow \cdots
\]
in which
\[
\phi_i(u) = (p_1(u), p_2(u)), \quad u \in H_i(X^*),
\psi_i(u_1, u_2) = g_1(u_1) = g_2(u_2), \quad u_1, u_2 \in H_i(X),
\]
where $p_i: X^* \to X$ is defined by $p_i(x_1, x_2) = x_i$ and $g_i: X \to (CX)^*$ is defined by $g_i(x) = ([x, 2 - i], [x, i - 1])$.

3. The examples. The 1-dimensional simplicial complexes $A_1$ and $A_2$ of Proposition 1.2 are drawn in Figure 3.1.

$H_*(A_1^*)$ is computed by starting with a simple closed curve and adding 1-simplices one at a time so that at each stage either Proposition 2.1 or 2.2 applies. Since $A_2$ consists of two copies of $K_{3,3}$ joined by a 1-simplex, $H_*(A_2^*)$ is easily computed using the remarks of §1 and Propositions 2.1
and 2.3. The result of these calculations is:

\[ \tilde{H}_j(A_1^*) \cong \tilde{H}_j(A_2^*) = 0 \quad \text{for } j \neq 1, 2, \]
\[ H_1(A_1^*) \cong H_1(A_2^*) \cong \mathbb{Z}^{25}, \]
\[ H_2(A_1^*) \cong H_2(A_2^*) \cong \mathbb{Z}^{34}; \]

we also have

\[ \tilde{H}_j(A_1) \cong \tilde{H}_j(A_2) = 0 \quad \text{for } j \neq 1, \quad H_1(A_1) \cong H_1(A_2) \cong \mathbb{Z}^8. \]

Since \( A_1 \) is planar while \( A_2 \) is nonplanar, this proves Proposition 1.2.

Now set \( B_1 = CA_1 \) and \( B_2 = CA_2 \). Then \( B_1 \) and \( B_2 \) are collapsible 2-dimensional simplicial complexes, \( B_1 \) is imbeddable in \( R^3 \) while \( B_2 \) is not. Using Proposition 2.4 we have

\[ \tilde{H}_j(B_1^*) \cong \tilde{H}_j(B_2^*) = 0 \quad \text{for } j \neq 2, 3, \]
\[ H_2(B_1^*) \cong H_2(B_2^*) \cong \mathbb{Z}^{36}, \]
\[ H_3(B_1^*) \cong H_3(B_2^*) \cong \mathbb{Z}^{24} \]

This proves Proposition 1.3.

Finally set \( C_1 = CF \) where \( F \) is the disjoint union of a point and a simple closed curve, and let \( C_2 = CK_5 \). Then using Proposition 2.4 we have

\[ H_2(C_1^*) \cong Z, \quad H_2(C_1^*) = 0, \]
\[ H_3(C_1^*) = 0, \quad H_3(C_2^*) \cong \mathbb{Z}. \]

Since both \( C_1 \) and \( C_2 \) are collapsible 2-dimensional nonplanar simplicial complexes, this proves Proposition 1.4.

\[ \text{Figure 3.1. The Complexes } A_1 \text{ (top) and } A_2. \]
4. **Proof of Theorem 1.5.** Suppose $X$ is a finite 1-dimensional simplicial complex such that $P(X^*)$ is a closed 2-manifold. We will show that either $X = K_5$ or $X = K_{3,3}$.

First observe that $X$ must be connected. For if $A$ and $B$ were distinct components of $X$, then $A \times B$ and $P(A^*)$ would be components of $P(X^*)$ and hence closed 2-manifolds. $A \times B$ could be a closed 2-manifold only if $A$ is a simple close curve, but then $P(A^*)$ would not be a 2-manifold.

Since every 1-cell of $P(X^*)$ must be a face of exactly two 2-cells of $P(X^*)$ we can conclude:

(i) $X$ contains no vertices of degree less than three or greater than four.

(ii) The closed star of a vertex of degree four in $X$ contains both vertices of every 1-simplex of $X$.

(iii) The closed star of a vertex of degree three in $X$ contains a vertex of every one simplex of $X$.

If $X$ contains a vertex of degree four, then (ii) implies that $X$ has exactly five vertices. So $X \subseteq K_5$ and hence $P(X^*) \subseteq P(K_5^*)$. Since $P(X^*)$ and $P(K_5^*)$ are closed manifolds we must have $P(X^*) = P(K_5^*)$ and hence $X = K_5$.

Finally, suppose every vertex of $X$ has degree three. By the preceding case, $X$ has more than five vertices. Choose a vertex $u_0$ of $X$ and let $u_1$, $u_2$, and $u_3$ be the other three vertices of the closed star of $u_0$ in $X$. If $w_1$ and $w_2$ are two other vertices of $X$, then (iii) implies that each 1-simplex meeting a $w_i$ must also meet some $u_j$, $j > 0$. This accounts for three 1-simplices meeting each of these six vertices, so $X$ contains no other 1-simplices or vertices. The 1-complex we have constructed is exactly $K_{3,3}$.

**References**


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