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LOCALIZATION IN A PRINCIPAL RIGHT IDEAL DOMAIN

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ABSTRACT. Let R be a principal right ideal domain with right D-chain $\{R^{(\alpha)} \mid 0 \leqq \alpha \leqq \delta\}$, and let $K_\alpha = R(R^{(\alpha)})^{-1}$ be the associated chain of quotient rings of R. The local skew degree of R is defined to be the least ordinal λ such that K_λ is a local ring. The main result states that for each $\alpha \geqq \lambda$, K_α is a local ring; equivalently, R has a unique $(\alpha+1)$-prime for $\delta > \alpha \geqq \lambda$.

In this note we consider conditions under which certain right quotient rings of a principal right ideal domain (PRI domain) are local rings. A PRI domain is a (not necessarily commutative) integral domain with unity in which each right ideal is a principal right ideal. By a local ring we understand a ring with unity in which the set of nonunits is closed under addition. Thus a local ring may be characterized as a ring with unity containing a unique maximal (right or left) ideal, namely the set of nonunits of the ring. A local PRI domain is then a PRI domain that has a unique prime (a nonzero nonunit that has no proper factorizations).

Before stating our main result we review some of the preliminaries that may be found in [1] and the references given there. A subset $S \neq \varnothing$ of nonzero elements of an integral domain R is a right quotient monoid in R if

(1) $ab \in S$ iff $a, b \in S$ $(a, b \in R)$,

(2) $a \in S$, $b \in R$ implies $aR \cap bS \neq \varnothing$.

Then the set $K = RS^{-1} = \{rs^{-1} \mid r \in R, s \in S\}$ can be made into a ring in the usual way. Addition and multiplication in K are carried out by using the fact that $a^{-1}b = b_1a_1^{-1}$ for each $a \in S$, $b \in R$ by (2). The ring K is the right quotient ring of R with respect to S;[1] it is an integral domain with unity with the property that the units of K that belong to R are precisely the members of S. It is easily established that K is a local ring iff $R \setminus S$ is an ideal of R. For each right ideal A of R, AS^{-1} is a right ideal of K; on the other hand, if B is a right ideal of K then $B = (B \cap R)S^{-1}$

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[1] In general, the right quotient ring of R with respect to S is defined with (1) replaced by the weaker condition that S be multiplicatively closed.

and $B \cap R$ is a right ideal of R. From this it follows that if R is a PRI domain then so is K.

For the remainder of this paper R is a PRI domain. Examples of right quotient monoids in R include the group U_R of units of R, and the monoid R^* of nonzero elements of R. Since prime factorizations in R are unique (up to order of factors and similarity) the dimension $\dim(a)$ of $a \in R^*$ can be defined to be $n \geqq 0$ if a is the product of n primes, and ∞ otherwise. It is shown in [1] that $R' = \{a \in R^* \mid \dim(a) < \infty\}$ is a right quotient monoid in R.

Any PRI domain R has a right D-chain. This means that there exists an ordinal δ and a chain $\{R^{(\alpha)} \mid 0 \leqq \alpha \leqq \delta\}$ of right quotient monoids in R satisfying the following conditions in which K_α denotes the right quotient ring $R(R^{(\alpha)})^{-1}$:

(1) $R^{(0)} = U_R$, $R^{(\delta)} = R^*$,

(2) $R^{(\alpha)} = \bigcup_{\beta<\alpha} R^{(\beta)}$ if α is a limit ordinal,

(3) $R^{(\alpha)} = (K_{\alpha-1})' \cap R$ if α is not a limit ordinal, $\alpha > 0$.

It follows that $R^{(\beta)} \subset R^{(\alpha)}$ and $K_\beta \subset K_\alpha$ if $\beta \leqq \alpha$. Also $K_0 = R$ and $K_\delta = R(R^*)^{-1}$ is the right quotient field of R. If I is an interval of ordinals, I^* will denote the set of nonlimit ordinals of I. For each $\alpha \in (0, \delta]^*$ an element $z \in R$ is called an α-prime if zR is maximal in $\{zR \mid z \in R \setminus R^{(\alpha-1)}\}$. Evidently, α-primes exist because of the acc for right ideals of R; 1-primes are the usual primes and if $\alpha > 1$, α-primes have infinite dimension in R and are primes in $K_{\alpha-1}$. The details regarding the construction of the D-chain may be found in [1].

We now define the *(right) local skew degree of R* to be the least ordinal λ such that K_λ is a local ring. This definition is valid since K_δ is itself a local ring.

Our main result is the following Theorem whose proof will follow from the two lemmas below.

THEOREM. *Let R be a PRI domain with right D-chain* $\{R^{(\alpha)} \mid 0 \leqq \alpha \leqq \delta\}$, *let* $K_\alpha = R(R^{(\alpha)})^{-1}$, *and let* λ *be the local skew degree of R. Then* K_α *is a local ring for each* $\alpha \geqq \lambda$. *Equivalently, R has a unique* $(\alpha + 1)$*-prime for* $\delta > \alpha \geqq \lambda$.

In what follows the term "unique" means unique up to right unit factor.

LEMMA 1. *Suppose R has a unique* β*-prime for some* $\beta \in [1, \delta]^*$. *Then R has a unique* α*-prime for each* $\alpha \in [\beta, \delta]^*$.

PROOF. Proceeding by transfinite induction let $\alpha \in (\beta, \delta]^*$ and assume R has a unique σ-prime whenever $\sigma \in [\beta, \alpha)^*$. To show R has a unique α-prime let y and z be two α-primes in R with $yR \neq zR$; we seek a contradiction. Let $dR = yR + zR$. We claim that if $\sigma \in [\beta, \alpha)^*$ then $d \in R \setminus R^{(\sigma)}$

For suppose $\sigma \in [\beta, \alpha)^*$ and let x be the unique σ-prime in R. Now $yR \subset xR$ by the maximum condition on the right ideals of R. In fact, $yR \subset \bigcap_{n=0}^{\infty} x^n R$; otherwise we would have $y = x^k s$ where k is the largest such integer and $s \in R \setminus xR$; it would then follow that $s \in R^{(\sigma-1)}$, otherwise $sR \subset xR$ by the maximum condition and therefore $y \in R^{(\sigma)}$ which contradicts the fact that y is an α-prime. Similarly we have $zR \subset \bigcap_{n=0}^{\infty} x^n R$ and therefore $dR \subset \bigcap_{n=0}^{\infty} x^n R$. Since x is prime in $K_{\sigma-1}$, the dimension of d in $K_{\sigma-1}$ is infinite; that is, $d \notin (K_{\sigma-1})'$ and so $d \in R \setminus R^{(\sigma)}$. This establishes the claim. Now $yR \subsetneqq dR$ implies $d \in R^{(\alpha-1)}$ because y is an α-prime. If $\alpha - 1$ is not a limit ordinal then we have contradicted the claim. If $\alpha - 1$ is a limit ordinal then $\beta < \alpha - 1$ because β is not a limit ordinal. Hence $R^{(\alpha-1)} = \bigcup_{\sigma<\alpha-1} R^{(\sigma)}$ and we may restrict the union to $\sigma \in [\beta, \alpha - 1)^*$. Thus we may choose $\sigma \in [\beta, \alpha - 1)^*$ such that $d \in R^{(\sigma)}$ which again contradicts the claim. We conclude $yR = zR$ which contradicts our original supposition.

LEMMA 2. *For each* $\alpha < \delta$, K_α *is a local ring iff* R *has a unique* $(\alpha + 1)$-*prime.*

PROOF. If K_α is a local ring then $R \setminus R^{(\alpha)}$ is an ideal, say $R \setminus R^{(\alpha)} = xR$. Then x is clearly the unique $(\alpha + 1)$-prime of R. Conversely, assume that R has a unique $(\alpha + 1)$-prime x, let z be any prime in K_α and choose $z_1 \in R$ such that $z_1 R = zK_\alpha \cap R$. Then $z_1 K_\alpha = zK_\alpha$ and $z_1 R \subset xR$ by the maximum condition for right ideals of R. Therefore $zK_\alpha \subset xK_\alpha$ which means $zK_\alpha = xK_\alpha$ since z and x are primes in K_α. This shows that K_α is a PRI domain with a unique prime. Consequently K_α is a local ring.

PROOF OF THE THEOREM. If the local skew degree λ of R is equal to δ then there is nothing to prove. Assume $\lambda < \delta$. Since K_λ is a local ring, R has a unique $(\lambda + 1)$-prime. If $\alpha > \lambda$, then R has a unique $(\alpha + 1)$-prime by Lemma 1 so that K_α is a local ring by Lemma 2.

In conclusion we refer the reader to **[1]** for examples of PRI domains with right D-chain of any preassigned length δ.

REFERENCES

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