A NOTE ON LIE-ADMISSIBLE NILALGEBRAS

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Abstract. It is shown that a finite dimensional, flexible, power-associative, Lie-admissible algebra \( \mathfrak{A} \) over a field of characteristic 0 is a nilalgebra if and only if there exists a Cartan subalgebra of \( \mathfrak{A} \) which is nil in \( \mathfrak{A} \).

Let \( \mathfrak{A} \) be a flexible algebra, that is, a nonassociative algebra satisfying the flexible law \((xy)x = x(yx)\). The algebra \( \mathfrak{A}^{-} \) is defined as the same vector space as \( \mathfrak{A} \) but with a multiplication given by \([x, y] = xy - yx\). Then \( \mathfrak{A} \) is said to be Lie-admissible if \( \mathfrak{A}^{-} \) is a Lie algebra. If \( \mathfrak{A} \) is finite dimensional, we consider a Cartan subalgebra \( \mathfrak{H} \) of \( \mathfrak{A}^{-} \). Since \( D_{h} \equiv R_{h} - L_{h} \) is a derivation of \( \mathfrak{A} \) for every \( h \) in \( \mathfrak{H} \) and \( \mathfrak{H} \) is the Fitting null component of \( \mathfrak{A}^{-} \) for \( D(\mathfrak{H}) \), it follows that \( \mathfrak{H} \) is a subalgebra of \( \mathfrak{A} \).

If \( \mathfrak{A}^{-} \) is a simple Lie algebra, it is shown that \( \mathfrak{A} \) is a nilalgebra ([3] and [4]). In case \( \mathfrak{A}^{-} \) is simple, the structure of \( \mathfrak{A} \) has been studied in [2] by using a Cartan subalgebra of \( \mathfrak{A}^{-} \) which is nil in \( \mathfrak{A} \). In this note we give a condition that \( \mathfrak{A} \) be a nilalgebra in terms of a Cartan subalgebra of \( \mathfrak{A}^{-} \).

Theorem. Suppose that \( \mathfrak{A} \) is a finite dimensional, flexible, power-associative, Lie-admissible algebra over a field of characteristic 0. Then \( \mathfrak{A} \) is a nilalgebra if and only if there exists a Cartan subalgebra of \( \mathfrak{A}^{-} \) which is nil in \( \mathfrak{A} \).

Proof. The “only if” part is obvious. Let \( \mathfrak{H} \) be a Cartan subalgebra of \( \mathfrak{A}^{-} \) which is nil in \( \mathfrak{A} \). Let \( \mathfrak{R} \) be the nil radical of \( \mathfrak{A} \) (the maximal nil ideal of \( \mathfrak{A} \)). Suppose that \( \mathfrak{A} \) is not a nilalgebra. Then the quotient algebra \( \mathfrak{A} = \mathfrak{A}/\mathfrak{R} \) satisfies the assumptions in the theorem and is semisimple. Since any flexible power-associative algebra of characteristic 0 is strictly power-associative, it follows from [4] that \( \mathfrak{A} \) has an identity \( \mathbb{I} \). Since the characteristic is 0, it also follows from [1, p. 379] that the homomorphic image \( \mathfrak{H} \) of \( \mathfrak{H} \) is a Cartan subalgebra of the Lie algebra \( \mathfrak{A}^{-} \). But then \( \mathbb{I} \) is in \( \mathfrak{H} \), and since \( \mathfrak{H} \) is nil in \( \mathfrak{A} \), this is a contradiction. Therefore \( \mathfrak{A} \) is a nilalgebra.
The center of $\mathfrak{U}^-$ is a subalgebra of $\mathfrak{U}$ and is contained in any Cartan subalgebra of $\mathfrak{U}^-$. The condition in the theorem cannot be relaxed to the case that the center of $\mathfrak{U}^-$ is nil in $\mathfrak{U}$.

Example. Let $\mathfrak{U}$ be an algebra with a basis $x, y, h, z$ over a field $\Phi$ of characteristic $\neq 2$ such that the multiplication is given by $xh = x$, $yh = \frac{1}{2}(x + 1)y$, $hy = \frac{1}{2}(1 - x)y$, $h^2 = h$ with $x \neq 0, 1$ in $\Phi$ and all other products are 0. It is shown that $\mathfrak{U}$ is flexible, power-associative and Lie-admissible, but not associative. Then $\mathfrak{U}z$ is the center of $\mathfrak{U}^-$ and $z^2 = 0$, while $\mathfrak{U}h + \mathfrak{U}z$ is a Cartan subalgebra of $\mathfrak{U}^-$ and is not nil in $\mathfrak{U}$.

References


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