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A NOTE ON LIE-ADMISSIBLE NILALGEBRAS

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ABSTRACT. It is shown that a finite dimensional, flexible, power-associative, Lie-admissible algebra $\mathfrak{A}$ over a field of characteristic 0 is a nilalgebra if and only if there exists a Cartan subalgebra of $\mathfrak{A}^-$ which is nil in $\mathfrak{A}$.

Let $\mathfrak{A}$ be a flexible algebra, that is, a nonassociative algebra satisfying the flexible law $(xy)x = x(yx)$. The algebra $\mathfrak{A}^-$ is defined as the same vector space as $\mathfrak{A}$ but with a multiplication given by $[x, y] = xy - yx$. Then $\mathfrak{A}$ is said to be Lie-admissible if $\mathfrak{A}^-$ is a Lie algebra. If $\mathfrak{A}$ is finite dimensional, we consider a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{A}^-$. Since $D_h \equiv R_h - L_h$ is a derivation of $\mathfrak{A}$ for every h in $\mathfrak{H}$ and $\mathfrak{H}$ is the Fitting null component of $\mathfrak{A}^-$ for $D(\mathfrak{H})$, it follows that $\mathfrak{H}$ is a subalgebra of $\mathfrak{A}$.

If $\mathfrak{A}^-$ is a simple Lie algebra, it is shown that $\mathfrak{A}$ is a nilalgebra ([**3**] and [**4**]). In case $\mathfrak{A}^-$ is simple, the structure of $\mathfrak{A}$ has been studied in [**2**] by using a Cartan subalgebra of $\mathfrak{A}^-$ which is nil in $\mathfrak{A}$. In this note we give a condition that $\mathfrak{A}$ be a nilalgebra in terms of a Cartan subalgebra of $\mathfrak{A}^-$.

THEOREM. *Suppose that $\mathfrak{A}$ is a finite dimensional, flexible, power-associative, Lie-admissible algebra over a field of characteristic* 0. *Then $\mathfrak{A}$ is a nilalgebra if and only if there exists a Cartan subalgebra of $\mathfrak{A}^-$ which is nil in $\mathfrak{A}$.*

PROOF. The "only if" part is obvious. Let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{A}^-$ which is nil in $\mathfrak{A}$. Let $\mathfrak{N}$ be the nil radical of $\mathfrak{A}$ (the maximal nil ideal of $\mathfrak{A}$). Suppose that $\mathfrak{A}$ is not a nilalgebra. Then the quotient algebra $\overline{\mathfrak{A}} = \mathfrak{A}/\mathfrak{N}$ satisfies the assumptions in the theorem and is semisimple. Since any flexible power-associative algebra of characteristic 0 is strictly power-associative, it follows from [**4**] that $\overline{\mathfrak{A}}$ has an identity $\bar{1}$. Since the characteristic is 0, it also follows from [**1**, p. 379] that the homomorphic image $\overline{\mathfrak{H}}$ of $\mathfrak{H}$ is a Cartan subalgebra of the Lie algebra $\overline{\mathfrak{A}}^-$. But then $\bar{1}$ is in $\overline{\mathfrak{H}}$, and since $\overline{\mathfrak{H}}$ is nil in $\overline{\mathfrak{A}}$, this is a contradiction. Therefore $\mathfrak{A}$ is a nilalgebra.

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The center of $\mathfrak{A}^-$ is a subalgebra of $\mathfrak{A}$ and is contained in any Cartan subalgebra of $\mathfrak{A}^-$. The condition in the theorem can not be relaxed to the case that the center of $\mathfrak{A}^-$ is nil in $\mathfrak{A}$.

EXAMPLE. Let $\mathfrak{A}$ be an algebra with a basis x, y, h, z over a field Φ of characteristic $\neq 2$ such that the multiplication is given by $xh = x$, $yh = \frac{1}{2}(\alpha + 1)y$, $hy = \frac{1}{2}(1 - \alpha)y$, $h^2 = h$ with $\alpha \neq 0, 1$ in Φ and all other products are 0. It is shown that $\mathfrak{A}$ is flexible, power-associative and Lie-admissible, but not associative. Then Φz is the center of $\mathfrak{A}^-$ and $z^2 = 0$, while $\Phi h + \Phi z$ is a Cartan subalgebra of $\mathfrak{A}^-$ and is not nil in $\mathfrak{A}$.

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