MORE ON THE SCHUR SUBGROUP

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Abstract. Let k be an abelian extension of the rational field Q. We show Schur's subgroup S(k) of the Bauer group B(k) is usually of infinite index. Generators for p-torsion elements of S(k) are found when k is the cyclotomic field of pth roots of unity.

Let k be an algebraic number field. We write B(k) for the Brauer group of k, and B_n(k) for the subgroup of B(k) generated by classes of division rings of exponent n. Let S(k) be the subgroup of B(k) consisting of all classes which contain a simple component of Q[G], the group algebra of a finite group G over the rational field Q. Following [6] we call S(k) the Schur subgroup of k. Let S_n(k) = S(k) \cap B_n(k). In [5] the structure of S_3(k) for k = Q(\sqrt[3]{-3}) is determined. Theorem 2 of this note generalizes the results of [5].

If A is a central simple algebra over k, we write [A] for the corresponding class in B(k).

Theorem 1. S_n(k) is of infinite index in B_n(k) for all n \geq 2 unless n = 2 and k = Q. In the exceptional case B_2(Q) = S_2(Q).

Proof. Since S(k) is trivial unless k is the field of an irreducible character of a finite group G, we may assume k/Q is abelian. Assume k \not= Q and r = [k:Q]. There are infinitely many primes of Q which split completely in k; let \pi_1, \pi_2, \ldots be an infinite list of them. For each \pi_i let \pi_i^{r_i} \ldots \pi_i^{r_m} be the primes of k lying over \pi_i. We construct [D_1], [D_2], \ldots, [D_m], \ldots in B_n(k) as follows:

D_m is the central division ring over k whose Hasse invariants satisfy:

\[ \text{inv} \ D_m = \frac{1}{n}, \quad \text{inv} \ D_m = -\frac{1}{n}, \]

\[ \text{inv} \ D_m = 0 \quad \text{at all other primes } \mathfrak{q} \text{ of } k. \]

The construction of the D_m is allowed by [1, Theorem 7.8]. By [2] we
have: \([D] \in S_n(k) \Rightarrow\) for each \(i\), \(D\) has constant index at \(g_i, \ldots, g_n\).
Hence \([D_1], \ldots, [D_m], \ldots\) above represent distinct cosets of \(S_n(k)\) in \(B_n(k)\). It follows that \([B_n(k):S_n(k)] = \infty\).

If \(k = \mathbb{Q}\), then \(S(k) = S_2(k)\) by the Brauer-Speiser theorem (see \([6]\)). The fact that \(B_2(\mathbb{Q}) = S_2(\mathbb{Q})\) follows from \([4]\).

Theorem 1 was also noted by Burton Fein.

Let \(p\) be a fixed odd prime. We will classify the algebras of index \(p\) in \(S(k)\), where \(k = \mathbb{Q}(\xi_p)\) is the cyclotomic field of \(p\)th roots of unity. This generalizes Theorem 2 of \([5]\).

If \(q\) is a prime, \(q \equiv 1 \pmod{p}\), then \(q\) splits completely in \(k = \mathbb{Q}(\xi_q)\). Let \(g_1, \ldots, g_{p-1}\) be the primes of \(k\) lying over \(q\). The field \(L = \mathbb{Q}(\xi_q, \xi_p)\) is cyclic over \(\mathbb{Q}(\xi_q)\) of degree \(q - 1\); let \(\tau\) be the generator of the Galois group of \(L\) over \(k\). Let \(H\) be the group generated by \(x, y,\) and \(z\) where \(x^p = y^q = 1, z\) acts on \(y\) according to the Galois action of \(\tau\) on \(Q(\xi_q)\), \(z^{p-1} = x\), and \(x\) is central in \(H\). Then the cyclic algebra \(A = (k(\xi_q), \tau, \xi_p)\) is a homomorphic image of \(Q[H]\), so \([A]\) is in the Schur subgroup of \(k\).

Clearly \([A]\) is a total matrix algebra since \(\xi_p^p = 1\); so \([A]\) has order 1 or \(p\) in \(B(k)\). \([A]\) has order 1 \(\Leftrightarrow\ \xi_p\) is a norm from \(L = k(\xi_q)\) to \(k\). We show \(\xi_p\) is not a local norm at the primes \(g_1, \ldots, g_{p-1}\) above. For convenience we fix \(g = g_1\).

The extension \(L/k\) is totally and tamely ramified at \(g\); let \(t\) be the unique prime of \(L\) lying over \(g\). If \(U_t\) (resp. \(U_{g}^1\)) denotes the units of \(L_t\) (resp. \(k_g\)) and \(U_{g}^1\) (resp. \(U_{g}^1\)) those which are 1 (mod \(t\)) (resp. 1 (mod \(g\))), then as in \([7, v, \#3]\) the norm induces a homomorphism:

\[N_0: U_t/U_{g}^1 \to U_{g}/U_{g}^1.\]

But \(U_t/U_{g}^1 \cong L_t^*\), the multiplicative group of the residue class field of \(L\) at \(t\). Similarly \(U_{g}/U_{g}^1 \cong k_g^* \cong L_t^* \cong Z_q^*\) as \(g\) is totally ramified in \(L\). Thus (1) reduces to a homomorphism:

\[N_0: Z_q^* \to Z_q^*\]

of cyclic groups of order \(q - 1\). By \([7, Proposition 5, p. 92]\) we have:
\(N_0(x) = x^{q-1}\) in (2). Hence the image of \(N_0\) is trivial, so \(N_0\) does not cover the image of \(\xi_p\); it follows that \(\xi_p\) is not a norm.

Thus \([A]\) represents an element of order \(p\) in \(S(k)\). Clearly \(A\) is split at all primes \(w, w \notin \{g_1, \ldots, g_{p-1}\}\), for each such prime is unramified from \(k\) to \(L\) and so \(\xi_p\) is a unit, hence a norm, at \(w\). By the proof of Theorem 5 of \([3]\) we have with suitable relabelling, invariants of \([A]\) of form \(1/p, 2/p, \ldots, (p - 1)/p\) at \(g_1, \ldots, g_{p-1}\).

We claim \(S_p(k)\) is generated by the classes \([A]\) above. Suppose \(D\) is a central division algebra over \(k\) with \([D] \in S_p(k)\); \(D\) has exponent \(p\). If \(p\) is a rational prime, \(q \equiv 1 \pmod{p}\), and \(g_1, \ldots, g_{p-1}\) are the primes of
1972 \[ k = Q(\xi_p) \] lying over \( q \), we have by [2], \( \text{inv}_{g_i} D = 0 \Rightarrow \text{inv}_v D = 0, i = 1, \ldots, p - 1 \). Assume \( \text{inv}_{g_i} D = a/p, (a, p) = 1 \). Set \( x = [\mathfrak{A}]^{-a} \cdot [D] \in S_p(k); \) then \( x \) has invariant 0 at \( g_i \Rightarrow \text{inv}_{g_i} x = 0 \) for \( i = 1, \ldots, p - 1 \Rightarrow \text{inv}_{g_i} D = ai/p = i(a/p) \). Thus \( D \) has invariants of type \( 1/p, 2/p, \ldots, (p - 1)/p \) at split primes over \( q \) where \( q \equiv 1 \pmod{p} \). We must show that \( D \) has no other non-0 invariants. By appropriate multiplication in \( B(k) \) as above we may assume \( \text{inv}_g D = 0 \) for all \( q \) lying over completely split primes of \( Q \).

\( D \) has no non-0 invariants at primes of \( k \) lying over odd rational primes by [8, Satz 10]. Also, since the index of \( D \) is \( p \) and \( p \neq 2 \), then \( D \) has no non-0 invariants at primes of \( k \) extending 2 by [8, Satz 11]. We have proved:

**Theorem 2.** If \( p \) is an odd prime, then the division rings \( D \) with \([D] \in S_p(k), k = Q(\xi_p) \) have invariants of type \( 1/p, 2/p, \ldots, (p - 1)/p \) at completely split primes of \( k \), and 0 everywhere else. The classes \([\mathfrak{A}] \) above generate \( S_p(k) \).

We note that Theorem 2 has the following unusual consequence: If \( p \) and \( q \) are distinct odd primes, then \( \xi_p \) is not a norm from \( Q(\xi_p, \xi_q) \) to \( Q(\xi_p) \Leftrightarrow q \equiv 1 \pmod{p} \).

Many thanks are due to both Burton Fein and Basil Gordon for pointing out to me the existence and applicability of [8]. Ken Fields has noted that there is no way of determining \( S_p(Q(\xi_p)) \) without first determining \( S_p(Q(\xi_p)) \).

**References**


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