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MINIMAL IDEALS IN GROUP RINGS

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ABSTRACT. Let $K[G]$ denote the group ring of G over an algebraically closed field K. In this paper we show that $K[G]$ has a minimal left ideal which affords a finite dimensional representation of the ring if and only if G is finite.

1. **Annihilator ideals.** Let R be a ring with one and let T be a subset of R. Then the left annihilator of T, $l_R(T)$, is defined by $l_R(T) = \{\alpha \in R \mid \alpha T = 0\}$. Clearly $l_R(T)$ is a left ideal of R. If in addition T is a left ideal of R, then $l_R(T)$ becomes an ideal (two-sided).

Let A be an ideal of R. We say that A is an annihilator ideal if $A \neq R$ and if $A = l_R(T)$ for some subset T of R. Observe that the condition $A \neq R$ is equivalent to $T \nsubseteq \{0\}$.

Let $K[G]$ denote the group ring of the group G over the field K. We do not assume unless otherwise stated that K is algebraically closed. If A is an annihilator ideal in $K[G]$, then we let φ_A denote the algebra homomorphism $\varphi_A : K[G] \to K[G]/A$. In this paper we study the K-algebra $K[G]/A$.

If $\alpha = \sum a_x x \in K[G]$, then we let Supp α be the finite subset of G given by Supp $\alpha = \{x \in G \mid a_x \neq 0\}$.

LEMMA 1.1. *Let A be an annihilator ideal in $K[G]$ and set $G_A = \{x \in G \mid \varphi_A(x) \in K\}$. Then G_A is a finite normal subgroup of G.*

PROOF. Since K is central in $K[G]/A$ it is clear that G_A is a normal subgroup of G. Let $A = l_{K[G]}(T)$ and let $\alpha \in T$, $\alpha \neq 0$. If $x \in G_A$ then there exists $k \in K$, $k \neq 0$, with $\varphi_A(x) = k = \varphi_A(k)$. Thus $x - k \in A$ so $(x - k)\alpha = 0$. This yields $x\alpha = k\alpha$ so $x(\text{Supp } \alpha) = \text{Supp } \alpha$. We have therefore shown that G_A permutes the finite nonempty set Supp $\alpha \subseteq G$ by left multiplication and hence G_A is finite.

The F.C. subgroup of G is defined by

$$\Delta = \Delta(G) = \{x \in G \mid [G : \mathbf{C}_G(x)] < \infty\}.$$

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We let $\theta : K[G] \to K[\Delta]$ denote the natural projection. The following is a simpler version of some work of M. Smith in [**2**].

THEOREM 1.2. *Let A be an annihilator ideal in $K[G]$. Let $\alpha \in K[G]$ and suppose that $\varphi_A(\alpha)$ is central in $K[G]/A$. Then $\alpha - \theta(\alpha) \in A$.*

PROOF. Say $A = l_{K[G]}(T)$ and let $\tau \in T$. Let $x \in G$. Then $\varphi_A(\alpha)$ commutes with $\varphi_A(x)$ so $\varphi_A(x\alpha - \alpha x) = 0$ and $x\alpha - \alpha x \in A$. Thus

$$1x(\alpha\tau) - \alpha x\tau = (x\alpha - \alpha x)\tau = 0.$$

Since this is true for all $x \in G$, Lemma 1.3 of [**1**] yields

$$(\alpha - \theta(\alpha))\tau = \theta(1)\alpha\tau - \theta(\alpha)\tau = 0.$$

Since this is true for all $\tau \in T$ we have $\alpha - \theta(\alpha) \in l_{K[G]}(T) = A$ and the result follows.

Let R be a ring with center Z. Then elements $\alpha_1, \alpha_2, \cdots, \alpha_n \in R$ are said to be linearly independent over Z if $\eta_1\alpha_1 + \eta_2\alpha_2 + \cdots + \eta_n\alpha_n = 0$ with $\eta_i \in Z$ implies that $\eta_1 = \eta_2 = \cdots = \eta_n = 0$.

COROLLARY 1.3. *Let A be an annihilator ideal in $K[G]$. Then*

(i) *$\varphi_A(K[\Delta(G)])$ contains the center of $K[G]/A$.*

(ii) *$\varphi_A(x_1), \varphi_A(x_2), \cdots, \varphi_A(x_n)$ are linearly independent over the center of $K[G]/A$ if $x_1, x_2, \cdots, x_n \in G$ are in distinct cosets of $\Delta(G)$.*

PROOF. (i) Suppose $\varphi_A(\alpha)$ is central in $K[G]/A$. Then by Theorem 1.2, $\alpha - \theta(\alpha) \in A$. Thus $\varphi_A(\alpha) = \varphi_A(\theta(\alpha))$.

(ii) Suppose $\bar{\eta}_1\varphi_A(x_1) + \bar{\eta}_2\varphi_A(x_2) + \cdots + \bar{\eta}_n\varphi_A(x_n) = 0$ with $\bar{\eta}_i$ in the center of $K[G]/A$. By part (i) there exists $\eta_i \in K[\Delta(G)]$ with $\bar{\eta}_i = \varphi_A(\eta_i)$. Thus the above becomes

$$\varphi_A(\eta_1 x_1 + \eta_2 x_2 + \cdots + \eta_n x_n) = 0.$$

Fix a subscript j and set

$$\begin{aligned} \alpha &= (\eta_1 x_1 + \eta_2 x_2 + \cdots + \eta_n x_n)x_j^{-1} \\ &= \eta_1 x_1 x_j^{-1} + \eta_2 x_2 x_j^{-1} + \cdots + \eta_n x_n x_j^{-1}. \end{aligned}$$

Then $\varphi_A(\alpha) = 0$ so $\varphi_A(\alpha)$ is central. Thus by Theorem 1.2 and the fact that the x_i's are in distinct cosets of $\Delta(G)$ we have $\alpha - \eta_j = \alpha - \theta(\alpha) \in A$. This yields $\bar{\eta}_j = \varphi_A(\eta_j) = \varphi_A(\alpha) = 0$ and the result follows.

LEMMA 1.4. *Let A be an annihilator ideal in $K[G]$ and let H be a subgroup of G. Then $B = A \cap K[H]$ is an annihilator ideal of $K[H]$ and therefore $\varphi_A(K[H]) = \varphi_B(K[H])$.*

PROOF. Clearly $B = A \cap K[H]$ is an ideal in $K[H]$. Let $\{x_\nu\}$ be a set of right coset representatives for H in G. Then every element $\alpha \in K[G]$

can be written uniquely as a finite sum $\alpha = \sum_\nu \alpha_\nu x_\nu$ with $\alpha_\nu \in K[H]$. If $\beta \in K[H]$ then clearly $\beta\alpha = 0$ if and only if $\beta\alpha_\nu = 0$ for all ν. Thus if $A = l_{K[G]}(T)$ then it follows easily that $B = l_{K[H]}(S)$ where $S = \{\tau_\nu \mid \tau \in T\} \subseteq K[H]$. This completes the proof.

2. **Finite dimensional algebras.** In this section we consider the possibility that $K[G]/A$ is a finite dimensional algebra over a field $F \supseteq K$.

LEMMA 2.1. *Let $G = \Delta(G)$ and let A be an annihilator ideal in $K[G]$. Suppose that $K[G]/A$ is a finite dimensional algebra over a field $F \supseteq K$. Then there exists a subgroup H of G with $[G:H] < \infty$ and $\varphi_A(K[H])$ central in $K[G]/A$.*

PROOF. Set $E = K[G]/A$ and for each $x \in G$ let $V_x = \mathbf{C}_E(\varphi_A(x))$. Then V_x is certainly an F-subspace of E. Since E is finite dimensional, it follows easily that there exists $x_1, x_2, \cdots, x_n \in G$ with

$$\bigcap_{x\in G} V_x = V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_n}.$$

Set $H = \mathbf{C}_G(x_1) \cap \mathbf{C}_G(x_2) \cap \cdots \cap \mathbf{C}_G(x_n)$. Since $G = \Delta(G)$ we have $[G:H] < \infty$.

Let $y \in H$. Then y centralizes $x_1, x_2, \cdots, x_n$ so $\varphi_A(y)$ centralizes $\varphi_A(x_1), \varphi_A(x_2), \cdots, \varphi_A(x_n)$ and hence

$$\varphi_A(y) \in V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_n} = \bigcap_{x\in G} V_x.$$

Thus for all $x \in G$, $\varphi_A(y)$ centralizes $\varphi_A(x)$ and the result follows.

LEMMA 2.2. *Let A be annihilator ideal in $K[G]$ and suppose that $K[G]/A$ is a finite dimensional algebra over a field $F \supseteq K$. Then there exists an annihilator ideal $B \supseteq A$ in $K[G]$ such that $K[G]/B$ is a finite dimensional simple algebra over F.*

PROOF. By induction on $\dim_F K[G]/A$. Certainly if this dimension is 1 then the result follows with $B = A$. Suppose first that $K[G]/A$ is prime. Then since it is a finite dimensional algebra, it follows from the Wedderburn theorems that $K[G]/A$ is simple and we need only take $B = A$.

Suppose now that $E = K[G]/A$ is not prime. Then there exists $\bar{\alpha}$, $\bar{\beta} \in E - \{0\}$ with $\bar{\alpha}E\bar{\beta} = 0$. Let $\bar{\alpha} = \varphi_A(\alpha)$, $\bar{\beta} = \varphi_A(\beta)$ and say $A = l_{K[G]}(T)$. Since $\bar{\beta} \neq 0$ we have $\beta T \neq 0$ and thus $S = K[G]\beta T \neq 0$. Then S is a left ideal in $K[G]$ so $C = l_{K[G]}(S)$ is an annihilator ideal in $K[G]$. Now A is an ideal so

$$AS = (AK[G]\beta)T \subseteq AT = 0$$

and hence $C \supseteq A$. Moreover, $\varphi_A(\alpha K[G]\beta) = 0$ so

$$\alpha S = (\alpha K[G]\beta)T \subseteq AT = 0$$

and $\alpha \in C$. Thus $C > A$ and $K[G]/C$ is a proper homomorphic image of E. This implies that $K[G]/C$ is an F-algebra and $\dim_F K[G]/C < \dim_F K[G]/A$. By induction there exists an annihilator ideal $B \supseteq C$ with $K[G]/B$ simple. Since $B \supseteq C \supseteq A$, the result follows.

LEMMA 2.3. *Let $x \in G$ have infinite order. Then $\bigcap_{n=1}^{\infty} (x^n - 1)K[G] = 0$.*

PROOF. Let $\{y_\nu\}$ be a set of right coset representatives for $\langle x\rangle$ in G. Then every element of $K[G]$ can be written uniquely as a finite sum $\alpha = \sum_\nu \alpha_\nu y_\nu$ with $\alpha_\nu \in K[\langle x\rangle]$. Thus $\alpha \in \bigcap_{n=1}^{\infty} (x^n - 1)K[G]$ if and only if $\alpha_\nu \in \bigcap_{n=1}^{\infty} (x^n - 1)K[\langle x\rangle]$ for all ν. Therefore it suffices to show that the latter intersection is zero or in other words we can assume that $G = \langle x\rangle$.

Let $\alpha \in K[G]$, $\alpha \neq 0$. Then Supp α is a finite subset of $\langle x\rangle$ and we let

$$\max(\alpha) = \max\{a \mid x^a \in \operatorname{Supp}\alpha\}, \qquad \min(\alpha) = \min\{a \mid x^a \in \operatorname{Supp}\alpha\}.$$

Now clearly if $\alpha \in (x^n - 1)K[G]$, $\alpha \neq 0$ then $\max(\alpha) - \min(\alpha) \geqq n$ and this therefore yields $\bigcap_{n=1}^{\infty} (x^n - 1)K[G] = 0$.

LEMMA 2.4. *Let A be an annihilator ideal in $K[G]$ and suppose that $K[G]/A$ is a field. Then G is a periodic group.*

PROOF. Suppose by way of contradiction that G has an element x of infinite order and let $A = l_{K(G)}(T)$. Then for each integer $n \geqq 1$, $x^n - 1$ is not a zero divisor in $K[G]$ by Lemma 2.4 of [**1**] so $x^n - 1 \notin A$ and $\varphi_A(x^n - 1) \neq 0$. Since $K[G]/A$ is a field there exists $\beta_n \in K[G]$ with $\varphi_A(x^n - 1)\varphi_A(\beta_n) = 1$. Thus $\varphi_A((x^n - 1)\beta_n - 1) = 0$. If $\tau \in T$ then this yields

$$(x^n - 1)\beta_n\tau - \tau = [(x^n - 1)\beta_n - 1]\tau = 0$$

so $\tau \in (x^n - 1)K[G]$. Since this is true for all $n \geqq 1$ and all $\tau \in T$ we have $T \subseteq \bigcap_{n=1}^{\infty} (x^n - 1)K[G] = 0$, by Lemma 2.3, a contradiction.

3. **Finite dimensional representations.** We now come to the main result of this paper.

THEOREM 3.1. *Let $K[G]$ denote the group ring of G over an algebraically closed field K and let A be an annihilator ideal in $K[G]$. If $K[G]/A$ is a finite dimensional algebra over a field $F \supseteq K$, then G is finite.*

PROOF. We first reduce the problem to $\Delta(G)$. By Corollary 1.3 (ii) we have $[G:\Delta] < \infty$. Also by Corollary 1.3 (i), $\varphi_A(K[\Delta])$ is a subring of $\varphi_A(K[G])$ containing the center. Thus $\varphi_A(K[\Delta]) \supseteq F$ so $\varphi_A(K[\Delta])$ is also a finite dimensional F-algebra. Finally by Lemma 1.4 there exists an annihilator ideal B in $K[\Delta]$ with $\varphi_A(K[\Delta]) = \varphi_B(K[\Delta])$. Therefore $K[\Delta]$ satisfies all the hypotheses of this theorem and since $[G:\Delta(G)] < \infty$ it suffices to consider the case $G = \Delta(G)$.

We now assume that $G = \Delta(G)$ and by Lemma 2.2 we can assume that A is so chosen that $K[G]/A$ is simple. Then $K[G]/A$ is a finite dimensional simple algebra so its center Z is a field. Now by Lemma 2.1, G has a subgroup H with $[G:H] = n < \infty$ and with $\varphi_A(K[H])$ central in $K[G]/A$. Thus $\varphi_A(K[H])$ is an integral domain but it is in fact a field. Let $x_1, x_2, \cdots, x_n$ be a set of right coset representatives for H in G. Then $K[G] = \sum_1^n K[H]x_i$ so $K[G]/A$ is a finitely generated module over $\varphi_A(K[H])$. Therefore every element of $K[G]/A$ is integral over $\varphi_A(K[H])$. In particular Z is a field which is integral over $\varphi_A(K[H]) \subseteq Z$ and this implies (by looking at the integral equation satisfied by the reciprocals of the elements of $\varphi_A(K[H]) - \{0\}$) that $\varphi_A(K[H])$ is also a field. By Lemma 1.4 there exists an annihilator ideal B in $K[H]$ with $\varphi_A(K[H]) = \varphi_B(K[H])$. Since $[G:H] < \infty$ it suffices to show that H is finite.

Thus we have reduced the problem to the case in which $K[G]/A$ is in fact a field. By Lemma 2.4, G is periodic. Now let $x \in G$. Then $x^n = 1$ for some $n > 1$ so $\varphi_A(x)$ is algebraic over K. Since K is algebraically closed this implies that $\varphi_A(x) \in K$. Thus in the notation of Lemma 1.1, $G = G_A$ and by that lemma G is finite. This completes the proof of the theorem.

Let E be an algebra over a field K. An irreducible representation ρ of E is said to be finite dimensional if $\rho(E)$ satisfies a polynomial identity over K.

COROLLARY 3.2. *Let $K[G]$ be the group ring of G over an algebraically closed field K. Then $K[G]$ has a minimal left ideal which affords a finite dimensional representation of the ring if and only if G is finite.*

PROOF. Suppose first that G is finite. Then $K[G]$ is a finite dimensional algebra so it has minimal left ideals. Let I be such an ideal. Then I affords a finite dimensional representation of $K[G]$ since $K[G]$ satisfies a polynomial identity.

Conversely suppose that $K[G]$ has a minimal left ideal I which affords a finite dimensional representation. Let A denote the kernel of the homomorphism $\rho: K[G] \to \text{End}(I)$. Then A is clearly the set of left annihilators of I so A is an annihilator ideal and $\rho = \varphi_A$. Now $\varphi_A(K[G])$ is a primitive ring satisfying a polynomial identity and hence by a theorem of Kaplansky (Theorem 6.4 of [1]) $\varphi_A(K[G])$ is a finite dimensional algebra over some field $F \supseteq K$. By Theorem 3.1, G is finite.

Finally we show by example that the above is false if K is not algebraically closed. Let $K = Q$ be the field of rationals and let G be the P_∞ group for some prime p. Thus

$$G = \langle x_1, x_2, x_3, \cdots, x_n, \cdots \mid x_1^p = 1, x_{n+1}^p = x_n \quad \text{for all } n \geqq 1\rangle.$$

Define p^nth roots of unity in the complex numbers C inductively by $\mathcal{E}_1$ is a primitive pth root of 1 and $\mathcal{E}_{n+1}^p = \mathcal{E}_n$. Then the map $\varphi: Q[G] \to C$

given by $x_i \to \mathcal{E}_i$ is clearly a homomorphism of $Q[G]$ onto a subfield F of C. We compute the kernel of this map.

Set

$$e = (1/p)(1 + x_1 + x_1^2 + \cdots + x_1^{p-1}).$$

Then e is an idempotent in $Q[G]$ and $\varphi(e) = 0$ since $1 + \mathcal{E}_1 + \mathcal{E}_1^2 + \cdots + \mathcal{E}_1^{p-1} = 0$. Thus the kernel of φ contains $eQ[G]$. Now let $\varphi(\alpha) = 0$. Then there exists an integer $n \geqq 1$ with $\alpha \in Q[\langle x_n \rangle]$ and we can write $\alpha = \sum_{i \geqq 0} a_i x_n^i$. Thus $0 = \varphi(\alpha) = \sum_{i \geqq 0} a_i \mathcal{E}_n^i$ and it follows that $\sum a_i x_n^i$, viewed as a polynomial in x_n, is divisible by the cyclotomic polynomial

$$1 + x_n^{p^{n-1}} + x_n^{2p^{n-1}} + \cdots + x_n^{(p-1)p^{n-1}} = pe.$$

Hence $\alpha \in eQ[G]$ and $eQ[G]$ is the kernel of φ.

Since $Q[G] = (1 - e)Q[G] + eQ[G]$ we see that $(1 - e)Q[G] \simeq F$. Thus $I = (1 - e)Q[G]$ is a minimal ideal in $Q[G]$ which affords a finite dimensional (in fact, commutative) representation of $Q[G]$. Since G is not finite, this yields the required counterexample.

References

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