

EQUATIONS IN THE THEORY OF MONADIC ALGEBRAS

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ABSTRACT. Identities in the theory of monadic algebras are equivalent to simpler "standard" ones. This property is used to prove some well-known results as well as to determine the minimum number of variables needed in an identity characterizing an equational class. Identities are also shown to be preserved under certain types of extension.

In this paper we prove that the equations of the theory of monadic algebras can be "reduced" to simpler "standard" ones. From this reduction, some properties already proved in [2] easily follow (see Theorem 3.3). We also obtain new results, mainly

(1) The determination of the minimum number of variables needed in an equation characterizing a given equational class. (See Theorem 3.4; this question is raised in [2].)

(2) A positive answer for the case $\alpha = 1$ to the following two questions: Does every equation holding in a CA_α also hold in its canonical embedding algebra? Does it also hold in its completion algebra? (See Corollary 3.6; these questions were raised by Monk at a seminar in algebraic logic at the University of Colorado.)

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1. Preliminaries. A *monadic algebra* \mathfrak{A} is considered as a structure $\langle A, +, \cdot, -, C \rangle$ where $\langle A, +, \cdot, - \rangle$ is a boolean algebra and C is a quantifier on $\langle A, +, \cdot, - \rangle$. CA_1 is the class of all monadic algebras.

The *first-order language* L_1 of CA_1 has v_0, v_1, \dots for individual variables but we use x_0, \dots, y_0, \dots , etc. to represent arbitrary variables; \cong is used for the formal equality; for the sake of simplicity we write $+$, \cdot , $-$, and C for the formal operation symbols corresponding to the concrete operations; 0 and 1 are taken as abbreviations for $v_0 \cdot -v_0$ and $v_0 + -v_0$

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respectively. The set of axioms characterizing CA_1 is denoted by Δ_1 and generally, when we write “ F and G are equivalent,” or “ τ is equal to σ ,” or “ $F \rightarrow G$,” or “ $\text{Mod } \mathcal{F}$ ” (the class of all models of \mathcal{F}), etc. we mean these notions relative to Δ_1 , i.e., $\Delta_1 \vdash F \leftrightarrow G$, $\Delta_1 \vdash \tau \simeq \sigma$, $\Delta_1 \vdash F \rightarrow G$, $\text{Mod } (\mathcal{F} \cup \Delta_1)$, etc. Other usual conventions will also be freely used.

For any term τ of L_1 , $(\zeta\tau)$ represents τ if $\zeta = 1$ and $-\tau$ if $\zeta = -1$; if $\zeta = 0$, $(\zeta C\tau)$ will simply denote τ .

A *boolean monomial* in the variables x_0, \dots, x_{n-1} ($n > 0$) is a term

$$(1) \quad (\zeta_0 x_0) \cdot (\zeta_1 x_1) \cdot \dots \cdot (\zeta_{n-1} x_{n-1})$$

where $\zeta \in {}^n\{1, -1\}$. The *sign* of x_j in (1) is ζ_j . We will assume that some numbering of the set of all boolean monomials in x_0, \dots, x_{n-1} has been fixed so that its members are $\beta_0, \dots, \beta_{n^*-1}$ (with $n^* = 2^n$).

A *monadic monomial* in the variables x_0, \dots, x_{n-1} ($n > 0$) is a term

$$(2) \quad (\varepsilon_0 C \beta_0) \cdot (\varepsilon_1 C \beta_1) \cdot \dots \cdot (\varepsilon_{n^*-1} C \beta_{n^*-1})$$

with β_i , $i < n^*$, the boolean monomials in x_0, \dots, x_{n-1} and $\varepsilon \in {}^{n^*}\{1, 0, -1\}$. (2) will be said to be *closed* if $\varepsilon \in {}^{n^*}\{1, -1\}$.

An *equation* in x_0, \dots, x_{n-1} ($n > 0$) is a formula $\tau \simeq \varphi$, where τ and φ are terms and x_0, \dots, x_{n-1} are the variables appearing in τ and φ . An *identity* with bound variables x_0, \dots, x_{n-1} is the universal closure of such an equation. *Standard equations* in x_0, \dots, x_{n-1} ($n > 0$) and *standard identities* with bound variables x_0, \dots, x_{n-1} ($n > 0$) will refer to formulas $\tau \simeq x_i \cdot -x_i$ ($i < n$), with τ a closed monadic monomial in x_0, \dots, x_{n-1} and to the universal closure of such a formula respectively. (We use $x_i \cdot -x_i$ instead of 0 to avoid the inessential distinction: v_0 occurs in τ , v_0 does not occur in τ .)

THEOREM 1.1. *Every term with variables occurring among x_0, \dots, x_{n-1} ($n > 0$) is equal to a finite sum of monadic monomials in x_0, \dots, x_{n-1} .*

PROOF. By induction.

THEOREM 1.2. *Every equation with variables occurring among x_0, \dots, x_{n-1} ($n > 0$) is equivalent to a finite conjunction of standard equations in x_0, \dots, x_{n-1} .*

PROOF. Essentially use Theorem 1.1, the normality and quasi-multiplicativity of C .

THEOREM 1.3. *Every identity is equivalent to a finite conjunction of standard identities having no more bound variables than it.*

PROOF. This is an immediate consequence of Theorem 1.2.

2. Standard identities. Theorem 1.3 reduces the study of identities to the study of standard identities. In this paragraph we establish two fundamental theorems concerning them.

Let F be a standard identity:

$$F = \forall x_0 \cdots \forall x_{n-1} (\mu \simeq x_i \cdot -x_i) \quad (i < n)$$

with $\mu = (\varepsilon_0 C \beta_0) \cdots (\varepsilon_{n^*-1} C \beta_{n^*-1})$, a closed monadic monomial in x_0, \dots, x_{n-1} . With F we associate a positive integer, $\text{Sgn } F$, the *signature* of F , defined by $\text{Sgn } F = \text{Card } (K)$ where $K = \{i < n^* \mid \varepsilon_i = 1\}$.

THEOREM 2.1. *Let F be a standard identity.*

- (1) *If $\text{Sgn } F = 0$, then $F \leftrightarrow 0 \simeq 0$.*
- (2) *If $\text{Sgn } F = p > 0$, then $F \leftrightarrow G$, for*

$$G = \forall y_0 \cdots \forall y_{p-1} \left(\bigwedge_{i < j < p} y_i \cdot y_j \simeq 0 \rightarrow Cy_0 \cdots Cy_{p-1} \simeq 0 \right).$$

PROOF OF CASE (1). In this case, $\mu = -C\beta_0 \cdots -C\beta_{n^*-1}$, hence

$$\mu \simeq -C \left(\sum_{i < n^*} \beta_i \right) \simeq -C1 \simeq 0.$$

PROOF OF CASE (2). We first prove " $G \rightarrow F$." Let $\mathfrak{A} = \langle A, +, \cdot, -, C \rangle$ be a monadic algebra model of G , and let $\bar{x}_0, \dots, \bar{x}_{n-1}$ be arbitrary elements of A . Consider a bijective function $f \in {}^p K$ (this is possible since $\text{Card } K = p$) and define for $i < p$, $\bar{y}_i = \bar{\beta}_{f(i)}$, where $\bar{\beta}_j$, $j < n^*$, is the element of A assigned to β_j under the valuation $x_k \rightarrow \bar{x}_k$ ($k < n$). Clearly the \bar{y}_i are disjoint. Since \mathfrak{A} is a model of G , we have:

$$C\bar{y}_0 \cdots C\bar{y}_{p-1} = 0, \quad C\bar{\beta}_{f(0)} \cdots C\bar{\beta}_{f(p-1)} = 0;$$

hence also

$$(\varepsilon_0 C \bar{\beta}_0) \cdots (\varepsilon_{n^*-1} C \bar{\beta}_{n^*-1}) = 0,$$

by the definition of f and usual boolean properties. This proves that \mathfrak{A} is also a model of F .

To prove " $F \rightarrow G$," observe that $H \rightarrow G$ where

$$H = \forall z_0 \cdots \forall z_{p-1} \left(\bigwedge_{i < j < p} z_i \cdot z_j \simeq 0 \wedge \sum_{i < p} z_i \simeq 1 \rightarrow Cz_0 \cdots Cz_{p-1} \simeq 0 \right).$$

Indeed, let \mathfrak{A} be a monadic algebra model of H and assume $\bar{y}_0, \dots, \bar{y}_{p-1}$ are disjoint elements of A . Let $\bar{z}_i = \bar{y}_i$ for $0 < i < p$ and $\bar{z}_0 = -\sum_{0 < i < p} \bar{y}_i$. Clearly the \bar{z}_i satisfy the hypothesis of H , hence $C\bar{z}_0 \cdots C\bar{z}_{p-1} = 0$. But, $C\bar{y}_i \leq C\bar{z}_i$ for $i < p$, hence $C\bar{y}_0 \cdots C\bar{y}_{p-1} = 0$, which proves that \mathfrak{A} is also a model of G .

Consequently, it suffices to show that $F \rightarrow H$. Let \mathfrak{A} be a monadic algebra model of F ; let also $\bar{z}_0, \dots, \bar{z}_{p-1}$ be disjoint elements of A the

sum of which is 1. Consider a bijective function $f \in {}^p K$ and define for $j < n^*$,

$$\bar{u}_j = \bar{z}_i \text{ for } j = f(i), \quad \bar{u}_j = 0 \text{ for } j \notin K.$$

Let, for every $i < n$, $I_i = \{k < n^* \mid \zeta_i^k = 1\}$ and $\bar{x}_i = \sum_{k \in I_i} \bar{u}_k$. (Here ζ_i^k is the sign of x_i in β_k .)

Since the \bar{u}_j are disjoint and their sum is 1,

$$\bar{x}_i \cdot \bar{x}_j = \sum_{k \in I_i \cap I_j} \bar{u}_k \text{ and } -\bar{x}_i = \sum_{k \in n^* \sim I_i} \bar{u}_k,$$

for any $i, j < n$.

These properties show that, for $j < n$,

$$\bar{\beta}_j = (\zeta_0^j \bar{x}_0) \cdots (\zeta_{n-1}^j \bar{x}_{n-1}) = \sum_{k \in J_j} \bar{u}_k$$

where $J_j = \bigcap_{i < n} (\zeta_i^j I_i)$. (By an obvious extension of a previous notation, (ζI_i) denotes I_i if $\zeta = 1$ and $n^* \sim I_i$ if $\zeta = -1$.)

But since $J_j = \{j\}$, it follows that $\bar{\beta}_j = \bar{u}_j$ ($j < n$). This result and the fact that \mathfrak{A} is a model of F show that

$$(\varepsilon_0 C \bar{u}_0) \cdots (\varepsilon_{n^*-1} C \bar{u}_{n^*-1}) = 0,$$

from which $C\bar{z}_0 \cdots C\bar{z}_{p-1} = 0$ follows by the normality of C and elementary boolean properties. This proves that \mathfrak{A} is also a model of H and that $F \rightarrow H$.

THEOREM 2.2. *Let F and G be standard identities. $F \rightarrow G$ iff $\text{Sgn } G = 0$ or $0 < \text{Sgn } F \leq \text{Sgn } G$.*

PROOF. If $\text{Sgn } G = 0$, then $F \rightarrow G$ for any F by Theorem 2.1 (1). If $0 < \text{Sgn } F \leq \text{Sgn } G$, Theorem 2.1 (2) clearly shows that $F \rightarrow G$. Conversely, suppose $0 < \text{Sgn } G < \text{Sgn } F$; it is easily seen that any finite simple monadic algebra with $\text{Sgn } F - 1$ atoms is a model of F but not of G . Consequently $F \not\rightarrow G$.

3. Some properties of identities and equational classes. In this paragraph we draw some consequences of the theorems of §2. For each $p < \omega$, choose F_p to be a standard identity with signature p and have the least possible number of variables among such formulas.

THEOREM 3.1. *If \mathcal{F} is a set of identities, there exists a unique $p < \omega$ such that $\text{Mod } (\mathcal{F}) = \text{Mod } (\{F_p\})$ and there exists $F \in \mathcal{F}$ such that*

- (1) $F \leftrightarrow F_p$,
- (2) F_p has no more bound variables than F .

PROOF. Let \mathcal{F}^* be the set of standard identities corresponding to the identities of \mathcal{F} by Theorem 1.3 and consider $\{\text{Sgn } F^* \mid F^* \in \mathcal{F}^* \text{ and } \text{Sgn } F^* > 0\}$. If this set is empty, we let $p = 0$ and if it is not, we let p be

its smallest element. In both cases we take for F an element of \mathcal{F} such that one of the standard identities, say F' , corresponding to F by 1.3 has signature p . By the definition of F_p and the choice of F' , F_p has no more bound variables than F' ; and by the choice of F , F' has no more bound variables than F ; this proves (2). The remaining statements of the theorem are reduced by 1.3 and 2.2 to trivial comparisons of signatures of standard identities.

COROLLARY 3.2. *Every identity is equivalent to a standard identity.*

In order to interpret Theorem 3.1 in terms of equational classes, we let $H_p = \text{Mod} (\{F_{p+1}\})$ for $p < \omega$ and $H_\omega = \text{Mod} (\{F_0\})$.

THEOREM 3.3 (MONK). (1) *The lattice of equational classes of monadic algebras is the chain: $H_0 \subset H_1 \subset \dots \subset H_\omega$.*

(2) *For every $p \leq \omega$, H_p is finitely axiomatizable.*

(3) *For every $p \leq \omega$, the equational theory of H_p is decidable.*

PROOF. For (1), apply 3.1 and 2.2; (2) is trivial. To prove (3), remark that the proofs of 1.1, 1.2, 1.3, and 3.1 together with Theorem 2.2 implicitly describe a decision method.

This theorem is essentially due to Monk (see [2]); it is readily seen that $H_\omega = CA_1$ and that for $p < \omega$, our H_p coincide with Monk's K_p (K_p is the equational closure of the class of simple finite monadic algebras with p atoms).

THEOREM 3.4. *For $0 < p < \omega$, the minimum number of bound variables needed in an identity characterizing H_p (relative to Δ_1) is the smallest n such that $2^n \geq p + 1$. For H_0 and H_ω , this number is 1.*

PROOF. For H_ω , take $\forall v_0$ ($0 \simeq 0$) and for H_0 , take $(\forall v_0)$ ($0 \simeq 1$). For $0 < p < \omega$, let n_0 denote the smallest n such that $2^n \geq p + 1$. The definitions of standard identity, of signature and of F_{p+1} show that F_{p+1} has exactly n_0 bound variables. On the other hand, if F is an identity characterizing H_p , Theorem 3.1 applied to $\mathcal{F} = \{F\}$ shows that F has at least as many variables as F_{p+1} , i.e., at least n_0 variables.

We now turn to the proof that identities are preserved under certain types of extension:

THEOREM 3.5. *Let $\mathfrak{A} = \langle A, +, \cdot, -, C \rangle$ be a monadic algebra and $\mathfrak{A}' = \langle A', +, \cdot, -, C' \rangle$ be a monadic algebra extending \mathfrak{A} and satisfying the following condition:*

(P) *There exists $B \subseteq A'$ such that*

(1) *for every $a \in A'$,*

$$C'a = \sum_{a \geq b \in B} \prod_{b \leq c \in A} Cc,$$

(2) (separation) for every $b, b' \in B$, $b \cdot b' = 0$, there exist $d, d' \in A$ such that

$$b \leq d, \quad b' \leq d' \quad \text{and} \quad d \cdot d' = 0.$$

If \mathfrak{A} is a model of an identity F , then \mathfrak{A}' is also a model of F .

PROOF. By 3.2, it suffices to show this for a standard F . If $\text{Sgn } F = 0$, apply Theorem 2.1 (1). Suppose now that $\text{Sgn } F = p > 0$. By part (1) of condition (P) and 2.1 (2) it suffices to show that if b_0, \dots, b_{p-1} are disjoint elements of B , then

$$(3) \quad \prod_{b_0 \leq c_0 \in A} \dots \prod_{b_{p-1} \leq c_{p-1} \in A} Cc_0 \dots Cc_{p-1} = 0.$$

The separation property (2) shows that we can choose disjoint $d_0, \dots, d_{p-1} \in A$ such that $d_i \geq b_i$, $i < p$. Consequently, the left member of (3) is smaller than or equal to

$$Cd_0 \dots Cd_{p-1}$$

and this element is 0 since \mathfrak{A} is a model of F (apply again 2.1 (2)).

For the notions of canonical embedding algebra and completion algebra, we refer to [1, §2.7] and [3] respectively.

COROLLARY 3.6. *Let \mathfrak{A} be a monadic algebra. If \mathfrak{A} is a model of an identity F , then the canonical embedding algebra of \mathfrak{A} and the completion algebra of \mathfrak{A} are also models of F .*

PROOF. Condition (P) of Theorem 3.5 is satisfied if we choose B as the set of all atoms of the canonical embedding algebra in the first case, and $B = A$ in the second case.

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