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A PRÜFER TRANSFORMATION FOR NONSELFADJOINT SYSTEMS

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ABSTRACT. The Prüfer transformation has been generalized to selfadjoint matrix differential equations by J. Barrett and others. Those results are extended to nonselfadjoint abstract systems of the form considered by Kamke in the scalar case and Reid in the matrix case.

Generalizations of the Prüfer transformation [1] to selfadjoint matrix differential systems of the form

$$Y' = B(x)Z; \quad Z' = -D(x)Y \qquad (a \leqq x < \infty) \tag{1}$$

have been studied by Barrett [2], Etgen [3], [4], and Reid [5], [6]. Hille [7] and Benson and the author [8] have established Prüfer transformations for abstract differential systems of the form (1) where $B(x)$, $D(x)$, $Y(x)$ and $Z(x)$ take their values in a Banach algebra $\mathcal{B}$. In all these studies the selfadjointness conditions $B(x) \equiv B^*(x)$; $D(x) \equiv D^*(x)$ play an essential role. The purpose of this note is to establish a Prüfer transformation for nonselfadjoint $\mathcal{B}$-valued matrix differential systems of the form

$$Y' = A(x)Y + B(x)Z, \qquad Z' = -D(x)Y + E(x)Z. \tag{2}$$

Such a transformation has been established in the scalar case by Kamke [9], [10], and (by different techniques) in the matrix case by Reid [6].

It is assumed that $A(x)$, $B(x)$, $D(x)$, and $E(x)$ are strongly continuous $\mathcal{B}$-valued functions on the semi-infinite interval $I = [a, \infty)$, and all derivatives below are to be taken in the strong topology. In passing from (1) to the more general system (2), we lose the relation $(Y^*Z - Z^*Y)' \equiv 0$ which is used in constructing "prepared" or "conjoined" solutions in the selfadjoint case. As pointed out to the author by W. T. Reid, the condition that a solution of (1) be conjoined is analogous to restricting oneself to real solutions in the scalar case. Accordingly our Prüfer transformation for (2) is motivated by a study by Barrett [11] of a Prüfer transformation for complex valued solutions of a scalar system of the form (1).

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The abstract trigonometric functions $s(x; a, Q)$, $C(x; a, Q)$, $S(x; a, Q)$, $c(x; a, Q)$ will be defined as solutions of a system of the form

$$
\begin{aligned}
&s' = Q(x)C; \quad s(a) = 0, \quad C' = -Q^*(x)s; \quad C(a) = E,\\
&S' = Q^*(x)c; \quad S(a) = 0, \quad c' = -Q(x)S; \quad c(a) = E.
\end{aligned}
\tag{3}
$$

Here $Q(x)$ is to be a strongly continuous $\mathfrak{B}$-valued function on I and E denotes the identity element in $\mathfrak{B}$. In case $Q(x)$ is nonsingular on I, $s(x; a, Q)$ and $c(x; a, Q)$ can be defined more directly as solutions of

$$(Q^{-1}(x)s')' + Q^*(x)s = 0, \qquad s(a) = 0; \quad s'(a) = Q(a),$$

and

$$(Q^{-1}(x)c')' + Q^*(x)c = 0, \qquad c(a) = E; \quad c'(a) = 0,$$

respectively.

We shall make use of the trigonometric identities contained in the following lemmas.

LEMMA 1. *If s, c, S, and C satisfy* (3) *on I, then*
(i) $s^*s + C^*C \equiv E$ *on I;*
(ii) $C^*S - s^*c \equiv 0$ *on I.*

PROOF. A direct calculation using (3) shows that $(s^*s + C^*C)' \equiv 0$ and $(C^*S - s^*c)' \equiv 0$ on I. The result now follows readily from the initial conditions of (3).

LEMMA 2. *If s, c, S, and C satisfy* (3) *on I, then*
(i) $ss^* + cc^* \equiv E$ *on I;*
(ii) $SS^* + CC^* \equiv E$ *on I;*
(iii) $Sc^* - Cs^* \equiv 0$ *on I;*
(iv) $sC^* - cS^* \equiv 0$ *on I.*

PROOF. Define $K = ss^* + cc^*$, $L = SS^* + CC^*$, $M = Sc^* - Cs^*$, and $N = sC^* - cS^*$. A direct calculation using (3) shows that K, L, M, and N satisfy the system

$$
\begin{aligned}
&K' = NQ^* - QM; \quad K(a) = E, \quad L' = MQ - Q^*N; \quad L(a) = E,\\
&M' = Q^*K - LQ^*; \quad M(a) = 0, \quad N' = QL - KQ; \quad N(a) = 0.
\end{aligned}
\tag{4}
$$

Since (4) has the obvious solution $K = L \equiv E$, $M = N \equiv 0$ and since solutions of (4) are unique, Lemma 2 is established.

Our Prüfer transformation will apply to a solution $Y(x)$, $Z(x)$ of (2) satisfying $Y(a) = 0$, $Z(a)$ nonsingular. We shall seek to represent such a solution in the form

$$Y(x) = s^*(x; a, Q)R(x); \qquad Z(x) = c^*(x; a, Q)R(x) \tag{5}$$

by proper choice of $Q(x)$ and $R(x)$.

If (5) is to be satisfied, then by (2)

$$s^*R' + C^*Q^*R = As^*R + Bc^*R, \quad c^*R' - S^*Q^*R = -Ds^*R + Ec^*R.$$

Using Lemma 2 this system can be solved for R' and Q^*R to yield

$$R' = (sAs^* + sBc^* - cDa^* + cEc^*)R; \qquad R(a) = Z(a), \tag{6}$$

$$Q^*R = (CAs^* + CBc^* + SDs^* - SEc^*)R. \tag{7}$$

Equation (7) suggests that $Q^*(x)$ should satisfy

$$Q^* = CAs^* + CBc^* + SDs^* - SEc^* \tag{7'}$$

or

$$Q = sA^*C^* + cB^*C^* + sD^*S^* - cE^*S^*. \tag{7''}$$

Thus equations (6) and (7″) are natural candidates for defining $Q(x)$ and $R(x)$ such that (5) is valid. A direct calculation shows that (6) and (7″) do indeed validate (5). However, since s, c, S, and C are all functions of Q, the solvability of (7″) remains to be established. This problem can be circumvented as in [**8**] by using (7′) and (7″) to define the trigonometric functions. Specifically, consider

$$\begin{aligned} s' &= (sA^*C^* + cB^*C^* + sD^*S^* - CE^*S^*)C; && s(a) = 0, \\ C' &= -(CAs^* + CBc^* + SDs^* - SEc^*)s; && C(a) = E, \\ S' &= (CAs^* + CBc^* + SDs^* - SEc^*)c; && S(a) = 0, \\ c' &= -(sA^*C^* + cB^*C^* + sD^*S^* - CE^*S^*)S; && c(a) = E. \end{aligned} \tag{3'}$$

By standard existence theory the system (3′) has a unique solution $s(x)$, $C(x)$, $S(x)$, $c(x)$ near $x = a$, and this solution can be continued as long as it remains bounded in the norm for $\mathfrak{B}$ (see [**8**]). Our principal result is as follows.

THEOREM 1. *Let* $Y(x)$, $Z(x)$ *be a solution of* (2) *satisfying* $Y(a) = 0$, $Z(a)$ *nonsingular. If* $R(x)$, *and* $s(x)$, $c(x)$ *are solutions of* (6) *and* (3′), *respectively, then* $Y(x)$, $Z(x)$ *admits the Prüfer transformation*

$$Y(x) = s^*(x)R(x); \qquad Z(x) = c^*(x)R(x). \tag{5'}$$

PROOF. It remains only to verify that the relations (5′) are compatible with (2). Using Lemma 1 we verify that

$$\begin{aligned} Y' &= s^*R' + s^{*\prime}R \\ &= s^*(sAs^* + sBc^* - cDs^* + cEc^*)R \\ &\quad + C^*(CAs^* + CBc^* + SDs^* - SEc^*)R \\ &= (s^*s + C^*C)(As^* + Bc^*)R + (C^*S - s^*c)(Ds^* - Ec^*)R \\ &= AY + BZ. \end{aligned}$$

An analogous calculation shows that $Z' = -DY + EZ$, and this completes the proof.

Theorem 1 establishes a local Prüfer transformation in the neighborhood of $x = a$. To show that this transformation is valid on $I = [a, \infty)$, it is only necessary to show that the solutions of (3′) remain bounded in $\mathfrak{B}$ for $a \leqq x < \infty$. The following is a slight generalization of an analogous result for selfadjoint systems [**8**].

THEOREM 2. *If $\mathfrak{B}$ is a C^*-algebra (i.e. an algebra of operators on a given Hilbert space $\mathfrak{H}$), then $|s(x)| \leqq 1$, $|C(x)| \leqq 1$, $|S(x)| \leqq 1$ and $|c(x)| \leqq 1$ for $a \leqq x < \infty$.*

PROOF. In the Hilbert space $\mathfrak{H}$ consider any solution $y(x)$, $z(x)$ of the system

$$y' = Q(x)z; \quad z' = -Q^*(x)y \qquad (a \leqq x < \infty) \tag{8}$$

satisfying $y(a) = 0$, $\|z(a)\| = 1$. Then

$$\frac{d}{dx}(\|y\|^2 + \|z\|^2) = (y, y') + (y', y) + (z, z') + (z', z)$$

$$= (y, Qz) + (Qz, y) - (z, Q^*y) - (Q^*y, z) = 0$$

so that

$$\|y(x)\|^2 + \|z(x)\|^2 \equiv 1$$

for $a \leqq x < \infty$. Now if $s(x)$, $C(x)$, $S(x)$, $c(x)$ satisfy (3), then for any constant unit vector e, $y(x) = s(x)e$, $z(x) = C(x)e$ satisfy (8) and

$$\|s(x)e\|^2 + \|C(x)e\|^2 \equiv 1.$$

This shows that $|s(x)| \leqq 1$ and $|C(x)| \leqq 1$ on I. An analogous argument shows that $|S(x)| \leqq 1$ and $|c(x)| \leqq 1$ on I.

COROLLARY. *If (2) is a real or complex matrix system, then a solution $Y(x)$, $Z(x)$ of (2) admits the Prüfer transformation (5′) on $I = [a, \infty)$.*

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