PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 1, January 1972

ON A TAUBERIAN THEOREM OF WIENER AND PITT

HAROLD G. DIAMOND[1]

ABSTRACT. N. Wiener and H. R. Pitt established a tauberian theorem which is "intermediate" between that of Wiener and Ikehara on one hand and a theorem of Hardy and Littlewood on the other. A new proof of the Wiener-Pitt theorem is given, using a technique of Bochner.

N. Wiener and H. R. Pitt established in [6] a tauberian theorem which is "intermediate" between that of Wiener and Ikehara on one hand and a theorem of Hardy and Littlewood on the other. Let $\alpha \in (0, 1)$ and $B > 0$. Let $\mathcal{C} = \mathcal{C}(\alpha, B)$ be the curve in the complex plane given by

$$\{\sigma + it : |t| = B\sigma^\alpha, 0 \leqq \sigma < \infty\}.$$

From a one sided boundedness condition on a function plus an L_1 hypothesis upon its Laplace transform along the curve $\mathcal{C}$, it is proved that the function has a small integral over certain intervals. A result of this type had been conjectured by J. Karamata and a special case ($\alpha = \frac{1}{2}$) was given by V. G. Avakumović [1].

The proof of Wiener and Pitt was quite intricate. Another version appears in Pitt's book [3, pp. 135–138], but that proof is valid if and only if $\alpha \geqq \frac{1}{2}$. This is so because Pitt assumes that

$$F(u) := \int_{-\infty}^{\infty} e^{itu} \exp(-|t|^{1/\alpha})\, dt > 0,$$

and this inequality is valid for all real u precisely when $\alpha \geqq \frac{1}{2}$ (cf. [2] and [4]). The object of the present paper is to give a shorter proof of the theorem. We shall use a Bochner type argument, integrating along line segments $\sigma + it$, $-\lambda \leqq t \leqq \lambda$, where we take $\sigma = L/x$, $\lambda = \frac{1}{2}B'\sigma^\alpha = \frac{1}{2}B'L^\alpha x^{-\alpha}$. The number L will be chosen later and will be independent of x, and $B' = \min(B, 1)$. We formulate the theorem substantially as in [3].

Received by the editors November 16, 1970.
AMS 1970 *subject classifications*. Primary 40E05.
Key words and phrases. Tauberian theorem, Laplace transform, Wiener-Ikehara theorem.
[1] Research supported by grant NSF GP 12363.

THEOREM. *Let s be a real measurable function that is supported in $(0, \infty)$, bounded from below, and satisfies $\int_0^\infty |s(y)| e^{-\sigma y}\, dy < \infty$ for all positive σ. For* Re $\omega > 0$, *define*

$$S(\omega) = \int_0^\infty e^{-\omega y} s(y)\, dy$$

and assume that

$$\lim_{\epsilon \to 0+} \int |S(\omega + \varepsilon) - S(\omega)|\, |d\omega| = 0,$$

where the integration is taken over any bounded arc of $\mathcal{C}$. Then for any $A > 0$,

$$\lim_{x \to \infty} \frac{1}{2Ax^\alpha} \int_{x - Ax^\alpha}^{x + Ax^\alpha} s(y)\, dy = 0.$$

Since s has no support near zero, $S(\omega)$ vanishes exponentially as Re $\omega \to \infty$. This fact plus the L_1 limit hypothesis guarantee both the absolute integrability of S along all of $\mathcal{C}$ and the validity of Cauchy's formula. Thus, for $|t| \leqq \lambda$ we can write

$$S(\sigma + it) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{S(z)\, dz}{z - \sigma - it}.$$

If we set

$$h_\lambda(y) = \lambda \left(\frac{\sin \lambda y/2}{\lambda y/2} \right)^2,$$

we have the following familiar Parseval formula (cf. [5, p. 84])

$$(*) \qquad \int_{-\lambda}^{\lambda} \left(1 - \frac{|t|}{\lambda}\right) e^{itx} S(\sigma + it)\, dt = \int_0^\infty h_\lambda(x - y) e^{-\sigma y} s(y)\, dy.$$

Let $\varphi(x)$ denote the left-hand side of (*) (recalling that σ and λ are functions of x). We begin by establishing a type of Riemann-Lebesgue lemma.

LEMMA 1.

$$\lim_{x \to \infty} \varphi(x) = 0.$$

PROOF. Expressing $S(\sigma + it)$ by the Cauchy formula and inverting the integration order, we have

$$\varphi(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} S(z) \left\{ \int_{-\lambda}^{\lambda} \frac{(1 - |t|/\lambda)}{z - \sigma - it} e^{itx}\, dt \right\} dz.$$

Let $\int_{-\lambda}^{\lambda}$ denote the inner integral, and integrate it by parts.

$$\left| \int_{-\lambda}^{\lambda} \right| = \left| \int_{-\lambda}^{\lambda} \frac{e^{itx}}{ix} \left\{ \frac{(1 - |t|/\lambda) i}{(z - \sigma - it)^2} - \frac{\operatorname{sgn} t}{\lambda(z - \sigma - it)} \right\} dt \right|$$

$$\leqq \frac{1}{x} \int_{-\lambda}^{\lambda} \frac{dt}{|z - \sigma - it|^2} + \frac{1}{\lambda x} \int_{-\lambda}^{\lambda} \frac{dt}{|z - \sigma - it|}.$$

Write $z = u + iv$. If $|v| \geqq \frac{3}{4}B'\sigma^\alpha$, then

$$|z - \sigma - it| \geqq |\mathrm{Im}\,(z - \sigma - it)| \geqq B'\sigma^\alpha/4$$

and $\int_{-\lambda}^{\lambda} = o(1)$ as $x \to \infty$. If $|v| < \frac{3}{4}B'\sigma^\alpha$, then $0 \leqq u < (3B'/4)^{1/\alpha}\sigma$ and

$$|z - \sigma - it|^2 = (u - \sigma)^2 + (v - t)^2 \geqq c^2\sigma^2 + (v - t)^2,$$

where $c^2 = 1 - (3B'/4)^{1/\alpha} > 0$. Thus, in this case

$$\left|\int_{-\lambda}^{\lambda}\right| \leqq \frac{1}{x}\int_{-\infty}^{\infty} \frac{dt}{c^2\sigma^2 + t^2} + \frac{1}{\lambda x}\frac{2\lambda}{c\sigma} = O(1).$$

Now express $\int_{\mathcal{C}}$ as the integral over $\mathcal{C}_1 = \{z \in \mathcal{C} : |\mathrm{Im}\, z| < \frac{3}{4}B'\sigma^\alpha\}$ plus the integral over $\mathcal{C}_2 = \mathcal{C} - \mathcal{C}_1$ and note that $\int_{\mathcal{C}_1} |S(z)|\,|dz| \to 0$ as $1/x$ (and hence σ) tends to 0, while the same integral over $\mathcal{C}_2$ is uniformly bounded. Thus $\varphi(x) \to 0$ as $x \to \infty$ □.

The next lemma shows s to be bounded "on the average." It is essential for the estimation of I_2 and I_4 at the conclusion of the article that the number β may be arbitrarily small and that the number c_1 not depend on β.

LEMMA 2. *There is a number c_1, which depends only on s, and for any number $\beta > 0$ there is a number X_β, which increases with $1/\beta$, such that, for all $x \geqq X_\beta$,*

$$(2\beta x^\alpha)^{-1}\int_{x-\beta x^\alpha}^{x+\beta x^\alpha} |s(y)|\,dy \leqq c_1.$$

PROOF. By the hypothesis of the theorem, $s(y) \geqq -k$ for some k and hence $|s(y)| \leqq s(y) + 2k$. We set $L = (2/B'\beta)^{1/\alpha}$ (just for the proof of this lemma) and have $\sigma = L/x$, $\lambda = \frac{1}{2}B'\sigma^\alpha = (\beta x^\alpha)^{-1}$. Then

$$2\left(\frac{\sin\frac{1}{2}}{\frac{1}{2}}\right)^2 \exp\{-\sigma x - \sigma/\lambda\}\frac{\lambda}{2}\int_{x-1/\lambda}^{x+1/\lambda} |s(y)|\,dy$$

$$\leqq \int_0^\infty h_\lambda(x - y)e^{-\sigma y}\{s(y) + 2k\}\,dy = \varphi(x) + 2k\int_0^\infty h_\lambda(x - y)e^{-\sigma y}\,dy$$

$$\leqq |\varphi(x)| + 2k\exp\{-\sigma x + \lambda e^L/\lambda\}\int_{|y-x|<e^L/\lambda} h_\lambda(x - y)\,dy$$

$$+ 2k\int_{|y-x|\geqq e^L/\lambda} h_\lambda(x - y)\,dy$$

$$\leqq |\varphi(x)| + 2k\exp\{-L + \beta L e^L x^{\alpha-1}\} + 16ke^{-L} < 20ke^{-L}$$

provided that x is large enough so that $|\varphi(x)| \leqq ke^{-L}$, $\exp\{\beta L e^L x^{\alpha-1}\} \leqq \frac{3}{2}$. Thus

$$\frac{\lambda}{2}\int_{x-1/\lambda}^{x+1/\lambda} |s(y)|\,dy \leqq \frac{20k}{8(\sin\frac{1}{2})^2}\exp\{\beta L x^{\alpha-1}\}$$

$$\leqq 12k =: c_1$$

provided x is large enough so that $\exp\{\beta L x^{\alpha-1}\} \leqq \frac{24}{5}(\sin \frac{1}{2})^2$.

Since βL and $e^L \to \infty$ with $1/\beta$ and φ is continuous and tending to zero, we can take X_β to be the infimum of all x for which each of the three inequalities is valid. □

COROLLARY 1. *There exists a number* c_2, *which depends only on* s, *such that, for all* $x \geqq 0$, $\int_0^x |s(y)|\,dy \leqq c_2 x$.

PROOF. The hypotheses of the theorem imply that $|s|$ is locally integrable and that s is zero near the origin. Now $x^{-1}\int_0^x |s(y)|\,dy$ is a continuous function on $(0, \infty)$ which is bounded at 0 and at ∞, and thus is bounded on $(0, \infty)$. □

COROLLARY 2. *Let* $\beta > 0$ *and suppose that* $w \geqq X_\beta$, *where* X_β *is as in the lemma. Suppose that* f *is a positive monotone function on* $[w, z]$ *and* $\frac{1}{2} \leqq f(x)/f(x + \beta x^\alpha) \leqq 2$ *for all* $x \in [w, z]$. *If* $z \geqq w + \beta w^\alpha$, *then*

$$\int_w^z f(u)\,|s(u)|\,du \leqq 4c_1 \int_w^z f(u)\,du.$$

PROOF. We may assume without loss of generality that f is increasing. If $z < (w + \beta w^\alpha) + \beta(w + \beta w^\alpha)^\alpha$, then

$$\int_w^z f(u)\,|s(u)|\,du \leqq f(z)\int_w^z |s(u)|\,du \leqq 4f(w)c_1(z - w)$$
$$\leqq 4c_1 \int_w^z f(u)\,du.$$

In the other case define a sequence $\{x_n\}_0^{N+1}$ by taking $x_0 = w$, $x_n = x_{n-1} + \beta x_{n-1}^\alpha$, $n = 1, 2, \cdots$, and taking x_{N+1} to be the largest number of the sequence not exceeding z. Apply the above inequalities to each of the intervals $[w, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_N, z]$. □

It is now convenient to approximate the right-hand side of (*) by a convolution. We show

LEMMA 3.

$$\lim_{x\to\infty} \int_0^\infty h_\lambda(x - y)s(y)\,dy = 0.$$

PROOF. Let $\epsilon > 0$ be given. Write

$$(h_\lambda * s)(x) = \int_0^\infty h_\lambda(x - y)s(y)\,dy = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = e^L \int_0^\infty h_\lambda(x - y)e^{-\sigma y}s(y)\,dy = e^L\varphi(x) = o(1)$$

as $x \to \infty$ and

$$\mathrm{II} = \int_0^\infty h_\lambda(x-y)s(y)\{1 - e^{L-\sigma y}\}\, dy.$$

Write

$$\mathrm{II} = \int_0^{x/2} + \int_{x/2}^{x-\nu x^\alpha} + \int_{x-\nu x^\alpha}^{x+\nu x^\alpha} + \int_{x+\nu x^\alpha}^\infty$$

where ν is a large positive number to be specified presently. By Corollary 1,

$$\left|\int_0^{x/2}\right| \leqq e^L \frac{4}{\lambda(x/2)^2} \cdot c_2 \frac{x}{2} = \frac{16e^L c_2}{B'L^\alpha x^{1-\alpha}} = o(1).$$

We can estimate the second and fourth integrals using Corollary 2. We first replace $h_\lambda(x-y)$ by $4/\lambda(x-y)^2 =: H(y)$, and note that H satisfies the monotonicity and slow growth conditions of the corollary on each of the two ranges. Thus

$$\left|\int_{x/2}^{x-\nu x^\alpha}\right| + \left|\int_{x+\nu x^\alpha}^\infty\right| \leqq 2 \cdot 4c_1 e^L \frac{4}{\lambda \nu x^\alpha} < 2\epsilon/5$$

provided that x is sufficiently large and $\nu > 160c_1e^L/B'L^\alpha\epsilon$. With this choice of ν, we estimate the third integral. If $x - \nu x^\alpha \leqq y \leqq x + \nu x^\alpha$ and x is sufficiently large, then

$$|e^{L-\sigma y} - 1| \leqq 2\,|L - \sigma y| \leqq 2L\nu x^{\alpha-1}.$$

Thus, by Lemma 2, we have

$$\left|\int_{x-\nu x^\alpha}^{x+\nu x^\alpha}\right| \leqq 2L\nu x^{\alpha-1}\lambda \int_{x-\nu x^\alpha}^{x+\nu x^\alpha} |s(y)|\, dy = O(x^{\alpha-1}\lambda x^\alpha) = o(1).$$

Now if x is sufficiently large, depending on s, L, $\mathcal{C}$ and ϵ, we have $|(h_\lambda * s)(x)| < \epsilon$. □

CONCLUSION OF THE ARGUMENT. Let χ_E be the indicator function of the set E. An easy estimate shows that for any positive number M we have

$$\begin{aligned} h_\lambda * \chi_{[-M,M]}(u) &= 1 + O\{\lambda^{-1}(M - |u|)^{-1}\}, && |u| < M, \\ &= O(1), && \text{always}, \\ &= O\{M\lambda^{-1}(|u| - M)^{-2}\}, && |u| > M. \end{aligned}$$

Here the constants implied by the O's are absolute. The preceding lemma implies that $((1/2M)\chi_{[-M,M]} * h_\lambda * s)(x) \to 0$ as $x \to \infty$, where M may tend to ∞ with x, so long as $x \geqq M$, say. We take $M = (A + \eta)x^\alpha$, where η is a positive number, presently to be specified, and write

$$\begin{aligned}(\chi_{[-M,M]} * h_\lambda * s)(x) &= \int_{-\infty}^{x} s(x-y)(\chi_{[-M,M]} * h_\lambda)(y)\,dy \\ &= \int_{-\infty}^{-(A+2\eta)x^\alpha} + \int_{-(A+2\eta)x^\alpha}^{-Ax^\alpha} + \int_{-Ax^\alpha}^{Ax^\alpha} \\ &\quad + \int_{Ax^\alpha}^{(A+2\eta)x^\alpha} + \int_{(A+2\eta)x^\alpha}^{x/2} + \int_{x/2}^{x} \\ &= \sum_{j=1}^{6} I_j, \quad \text{say.}\end{aligned}$$

Now

$$I_3 = \int_{x-Ax^\alpha}^{x+Ax^\alpha} s(y)\,dy + O\left(\frac{c_1 Ax^\alpha}{\lambda\eta x^\alpha}\right).$$

$I_2 + I_4 = O(c_1\eta x^\alpha)$, $I_1 + I_5 = O(c_1 M/\lambda\eta x^\alpha)$, by Corollary 2, and $I_6 = O(Mc_2/\lambda x)$.

The four O's are absolute and the estimates are valid for all sufficiently large x. Thus we have

$$\begin{aligned}&\left|\frac{1}{2Ax^\alpha}\int_{x-Ax^\alpha}^{x+Ax^\alpha} s(y)\,dy\right| \\ &\leqq c_3\left\{\frac{c_1}{\eta B'L^\alpha} + \frac{c_1\eta}{A} + \frac{c_1(A+\eta)}{AB'L^\alpha\eta} + \frac{c_2(A+\eta)x^{\alpha-1}}{AB'L^\alpha} + o(1)\right\},\end{aligned}$$

where c_3 is absolute.

Given ϵ in $(0, 1)$, take η so small that $c_3c_1\eta/A < \epsilon/5$. Next, take L sufficiently large that

$$(A+\eta)c_1c_3/(AB'L^\alpha\eta) < \epsilon/5.$$

Then take x so large that all the preceding inequalities are valid and

$$c_3c_2(A+\eta)x^{\alpha-1}/(AB'L^\alpha) + c_3 o(1) < 2\epsilon/5.$$

We conclude then that

$$\left|\frac{1}{2Ax^\alpha}\int_{x-Ax^\alpha}^{x+Ax^\alpha} s(y)\,dy\right| < \epsilon$$

for all sufficiently large x, and the proof of the theorem is complete. □

Possibly the L_1 limit in the hypothesis of the theorem can be relaxed to the L_1 bound $\int_{\mathcal{C}} |S(\omega)|\,|d\omega| < \infty$. One would then have to show (if possible) that the singularity of S at zero, when approached from "within" $\mathcal{C}$, was sufficiently weak to permit the application of Cauchy's theorem.

REFERENCES

1. V. G. Avakumović, *Théorèmes relatifs aux intégrales de Laplace sur leur frontière de convergence*, C. R. Acad. Sci. Paris **204** (1937), 224–226.

2. P. Lévy, *Sur une application de la dérivée d'ordre non entier au calcul des probabilités*, C. R. Acad. Sci. Paris **176** (1923), 1118–1120.

3. H. R. Pitt, *Tauberian theorems*, Tata Institute of Fundamental Research Monographs on Math. and Physics, no. 2, Oxford Univ. Press, London, 1958. MR **21** #5109.

4. G. Pólya, *On the zeros of an integral function represented by Fourier's integral*, Messenger Math. **52** (1923), 185–188.

5. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford Univ. Press, London, 1937.

6. N. Wiener and H. R. Pitt, *A generalization of Ikehara's theorem*, J. Math. Phys. **17** (1939), 247–258.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801