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GLOBAL HYPOELLIPTICITY AND LIOUVILLE NUMBERS[1]

STEPHEN J. GREENFIELD AND NOLAN R. WALLACH

ABSTRACT. We consider global hypoellipticity of constant coefficient differential operators on the 2-torus, and prove that it is equivalent to an algebraic growth condition on the symbol. This is applied to give necessary and sufficient conditions that a constant coefficient vector field be globally hypoelliptic. Similar results are true on compact homogeneous spaces.

Let $T^2 = \{(\exp i\theta_1, \exp i\theta_2);\ \theta_j \in \boldsymbol{R}\}$. If $L \in \mathfrak{D}'(T^2)$ (a distribution on T^2) define $\hat{L}(n, m) = L(\exp(-in\theta_1 - im\theta_2))$ (see the normalization below for functions). $\hat{L}$ is a function on $\boldsymbol{Z} \times \boldsymbol{Z}$, the *Fourier transform* of L. We write $L \sim \{\hat{L}(n, m)\}$ to indicate the correspondence. A function $T: \boldsymbol{Z} \times \boldsymbol{Z} \to \boldsymbol{C}$ is the Fourier transform of a distribution L iff there is $K > 0$ so that $|T(n, m)| \leqq K(n^2 + m^2 + 1)^K$. ($T$ is of polynomial growth ([S, Chapter 7]).) L can be reconstructed from T by:

$$L = \sum_{n,m} T(n, m) \exp(in\theta_1 + im\theta_2).$$

(This implies the normalization

$$f(g) = (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) g(x, y)\, dx\, dy$$

for f a function.)

Suppose P is an invariant differential operator on T^2:

$$P = \sum_{k,l=0}^{N} c_{kl} D_1^k D_2^l,$$

where $c_{kl} \in \boldsymbol{C}$ and $D_j^k = (i^{-1}\, \partial/\partial\theta_j)^k$. Define $\hat{P}(n, m) = \sum_{k,l=0}^{N} c_{kl} n^k m^l$.

We say P is globally hypoelliptic on T^2 when:

(GH) If $g \in C^\infty(T^2)$, and $Pf = g$, $f \in \mathfrak{D}'(T^2)$, then $f \in C^\infty(T^2)$.

THEOREM. *P is* (GH) *if and only if there are positive real numbers L, M so that*:

(LM) $|\hat{P}(n, m)| \geqq L/(n^2 + m^2)^M$, *for* $|n|$, $|m|$ *sufficiently large.*

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PROOF. When $h \in \mathfrak{D}'(T^2)$, then $h \in C^\infty(T^2)$ iff

$$(*) \qquad \sup \frac{|\hat{h}(n, m)|}{(n^2 + m^2 + 1)^K} < +\infty \quad \text{for each } K.$$

(This is equivalent to the perhaps better known condition

$$\sum_{|n|+|m|>0} |\hat{h}(n, m)|^2 (n^2 + m^2)^K < \infty.)$$

Suppose now $Pf = g$, f, $g \in \mathfrak{D}'(T^2)$, and $f \sim \{a_{nm}\}$, and $g \sim \{b_{mn}\}$. Then $\hat{P}(n, m) a_{nm} = b_{nm}$.

(LM) $\Rightarrow$ (GH): By (LM), $\hat{P}(n, m) \neq 0$ if $|n| + |m|$ is sufficiently large. Thus $a_{nm} = (b_{nm}/\hat{P}(n, m))$, $|n| + |m|$ large. Since

$$1/|\hat{P}(n, m)| \leqq (n^2 + m^2)^M / L,$$

(*) for $\{b_{nm}\}$ gives (*) for $\{a_{nm}\}$.

$\sim$(LM) $\Rightarrow$ $\sim$(GH): If (LM) is false, there is a sequence $\{(n_j; m_j)\} \subseteq \boldsymbol{Z} \times \boldsymbol{Z}$ so that $(n_j; m_j) \to +\infty$ and $|\hat{P}(n_j; m_j)| \leqq 1/(n_j^2 + m_j^2)^j$. Put $f = \sum_j \exp(in_j\theta_1 + im_j\theta_2)$. Then $f \in \mathfrak{D}'(T^2) - C^\infty(T^2)$, but $Pf \in C^\infty(T^2)$.

REMARK. If P has order N, $|\hat{P}(n, m)| \leqq L(n^2 + m^2)^{N/2}$. If P is elliptic, Gårding's inequality implies that we can take $M = -N/2$ in (LM). We do not get local hypoellipticity on T^2 since we only consider simple behavior of the *real* Fourier transform $\hat{P}$ (see the criterion for hypoellipticity on $\boldsymbol{R}^n$ given in **[H]**. See also **[B]**).

COROLLARY. *If P is* (GH), *then $P: C^\infty(T^2) \to C^\infty(T^2)$ is Fredholm of index* 0.

PROOF. P is continuous. (LM) implies that $\{\hat{P}(n, m)\}$ is almost never 0. Put $S = \{(n, m) \mid \hat{P}(n, m) = 0\}$. S is finite. If $f \in C^\infty(T^2)$, and $f \sim \{a_{nm}\}$, let $\pi_S f$ be the C^∞ function defined by $\{d_{nm}\}$, with $d_{nm} = a_{nm}$ for $(n, m) \in S$, and $d_{n,m} = 0$ otherwise. Since S is finite π_S is a continuous projection.

$\ker P = \{f \in C^\infty(T^2): (I - \pi_S)f = 0\}$. If $Pf = g$, and $g \in C^\infty(T^2)$ with $\pi_S(g) = 0$, then there is $f \in C^\infty(T^2)$ so that $Pf = g$ (obtain f by Fourier transform from g-condition (LM) guarantees solvability). Thus dim ker P = dim coker P = cardinality of S.

Condition (LM) is rather particular. Suppose $P = D_1 + cD_2$, $c = a + ib \in \boldsymbol{C}$. If $b \neq 0$, P is elliptic and surely (GH). Suppose $b = 0$, and $a \neq 0$.

If $a = R/S$ (R, S integers), then $\hat{P}(-Rt, St) = 0$. Thus $\hat{P}$ has infinitely many zeros, and P is not (GH).

PROPOSITION. *Suppose α is a real irrational number. The vector field $P = D_1 - \alpha D_2$ is globally hypoelliptic if and only if α is not a Liouville number.*

PROOF. We recall (see [HW]) that $\alpha \in \boldsymbol{R}$ is a Liouville number if it can be approximated by rationals to any order. That is, for every positive integer N, there is $K > 0$, and infinitely many integer pairs (n, m) so that: (**) $|\alpha - n/m| < K/m^N$.

Condition (LM) for P becomes: $|n - \alpha m| \geqq L/(n^2 + m^2)^M$. Or $|\alpha - n/m| \geqq L/(n^2 + m^2)^M m$. By adjusting L we can suppose $L/(n^2 + m^2)^M m < 1$ for $|n| + |m| > 0$. We need only consider $|\alpha - n/m| < 1$. So n and m have the same order, and condition (LM) (with suitable change of L) becomes: $|\alpha - n/m| \geqq L/m^{2M+1}$, for some $M, L > 0$.

It is now clear that (LM) is equivalent to α not a Liouville number.

REMARK. Thus we have vector fields which are globally hypoelliptic but (since they are *real* vector fields) certainly not locally hypoelliptic. They are connected with the closed divergences of C. S. Herz [Z].

BIBLIOGRAPHY

B. R. Strichartz, *Invariant pseudo-differential operators on a Lie group* (to appear).

F. A. Cerezo and R. Rouvière, *Solution élémentaire d'un opérateur différential linéaire invariant à gauche sur un groupe de Lie réal compact et sur un espace homogène réductif compact*, Ann. Sci. École. Norm. Sup. (4) **2** (1969), 561–581.

H. L. Hörmander, *Linear partial differential operators*, Die Grundlehren der math. Wissenschaften, Band 116, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #4221.

HW. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London, 1960. MR **16**, 673.

S. L. Schwartz, *Théorie des distributions*. Tomes I, II, Actualités Sci. Indust., nos. 1091, 1122, Hermann, Paris, 1950, 1951. MR **12**, 31; 833.

W. Nolan R. Wallach, *Differential operators on homogeneous spaces*, Marcel Dekker, New York (to appear).

Z. C. Z. Herz, *Functions which are divergences*, Amer. J. Math. **92** (1970), 641–656.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903