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A NEW l_1 ESTIMATE AND A PROBLEM OF KATZNELSON[1]

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ABSTRACT. We produce a new estimate on the l_1 norm in terms of the closeness of approximation by functions with small derivatives. We thereby solve a problem of Katznelson and give new proofs of some old l_1 estimates.

We define, as usual, the l_1 norm by $\|f(x)\| = \sum_{-\infty}^{\infty} |c_n|$ where c_n are the fourier coefficients of $f(x)$. Many estimates of $\|f\|$ are available in terms of the smoothness of f. Notably we have Carlson's inequality

$$(1) \qquad \|f\| \leqq |c_0| + A\left(\int_{-\pi}^{\pi} |f(x)|^2\,dx \int_{-\pi}^{\pi} |f'(x)|^2\,dx\right)^{1/4}$$

which bounds the norm in terms of the size of f and its first derivative.

Recently, I. Katznelson asked for an estimate on the norm of $((e^{ix} - c)/(1 - \bar{c}e^{ix}))^n$, $|c| < 1$, n a positive integer, It is known that this norm is exactly of the order $\sqrt{n}$ for any fixed $c \neq 0$ and Katznelson's specific question was whether this bound is uniform in c.

A direct application of (1) gives an upper bound which tends to ∞ as $|c| \to 1$. But this is due to a contribution to $\int |f'|^2$ on a very small interval wherein f' is large and the question arises as to whether we can give a better bound than (1) if f' is "usually" small.

A glance at the "triangle function" and the "trapezoid function" give respectively the optimistic and pessimistic answers to this question. Both have their large derivative restricted to very small intervals but the first has a uniformly bounded norm while the second does not. The difference between the behavior of these two functions, however, is that the high derivative changes the function only in a small interval in the first case whereas it changes the function on a long interval in the second case.

Katznelson's function $((e^{ix} - c)/(1 - \bar{c}e^{ix}))^n$ is like the triangle function in this respect. For $|c|$ near 1 it is virtually a constant except for a small interval about arg c and so one feels that an improvement of (1) is possible here. Indeed this is exactly what we achieve in this note.

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To formulate our theorem let us imagine absolutely continuous period 2π functions $f_\delta(x)$, $0 \leqq \delta \leqq 1$, which are "close to" $f(x)$ for δ near 0 but which have "reasonably small" derivatives for δ away from 0. It turns out that the appropriate measure for the terms "close to" and "reasonably small" is that of L^2, and that the existence of such smooth approximations to f does give an effective bound on its norm. The exact details follow:

THEOREM. *Suppose that*

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-f_\delta(x)|^2\,dx\right)^{1/4} \leqq F(\delta), \qquad \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f_\delta'(x)|^2\,dx\right)^{1/4} \leqq G(\delta),$$

where $F(\delta)$ increases continuously from 0 *and $G(\delta)$ decreases continuously to* 0 *as δ goes from* 0 *to* 1. *Then*

$$\|f\| \leqq |c_0| + A\int_0^1 G(\delta)\,dF(\delta).$$

REMARKS. That our theorem contains Carlson's inequality, (1), can be seen by simply choosing $f_\delta(x) = f(x)(1-\delta)$. We may also obtain the affirmative answer to Katznelson's question by choosing, in that case, $f_\delta(x) = f(x)$ for $|x - \arg c| \geqq \delta\pi$ and $f_\delta(x)$ linear for $|x - \arg c| \leqq \delta\pi$. An easy computation gives $F(\delta) = A\delta^{1/4}$,

$$G(\delta) = A(n(1-|c|))^{1/2}\min\left((1-|c|)^{-3/4}, \delta^{-3/4} - 1\right)$$

and we get the

COROLLARY. $\|((e^{ix}-c)/(1-\bar{c}e^{ix}))^n\| \leqq A\sqrt{n}$, *$A$ independent of c and n.*

As an additional bonus we obtain Bernstein's theorem that a function in Lip (α), $1 > \alpha > \frac{1}{2}$, has finite norm. Here we choose $f_\delta(x) = \delta^{-1}\int_x^{x+2\pi\delta} f(t)\,dt$ which yields the estimates $F(\delta) = A\delta^{\alpha/2}$, $G(\delta) = A[\delta^{(\alpha-1)/2} - 1]$ and the result follows immediately.

PROOF OF THE THEOREM. We may assume that $c_0 = 0$. Call $f_\delta(x) = \sum_{-\infty}^{\infty} c_{n,\delta}e^{inx}$ and write $T = (G(\delta)/F(\delta))^2$. We have, by Schwarz's inequality and Parseval's theorem,

$$\begin{aligned}\sum_{T<|n|<2T}|c_n| &\leqq \sum_{T<|n|<2T}|c_{n,\delta}| + \sum_{T<|n|<2T}|c_n - c_{n,\delta}| \\ &\leqq \left(\sum_{T<|n|<2T}\frac{1}{n^2}\right)^{1/2}\left(\sum n^2|c_{n,\delta}|^2\right)^{1/2} \\ &\quad + (2T+1)^{1/2}\left(\sum |c_n - c_{n,\delta}|^2\right)^{1/2} \\ &\leqq \left(\frac{2}{T}\right)^{1/2}G^2(\delta) + (2T)^{1/2}F^2(\delta) = 2\sqrt{2}\,F(\delta)G(\delta).\end{aligned}$$

Also $dT/T = 2\,dG(\delta)/G(\delta) - 2\,dF(\delta)/F(\delta)$, and T goes from 0 to ∞ as δ goes from 1 to 0 so that

$$\int_0^\infty \sum_{T<|n|<2T} |c_n| \frac{dT}{T} \leqq 4\sqrt{2}\int_0^1 G(\delta)\,dF(\delta) - F(\delta)\,dG(\delta).$$

Since moreover, $\int_0^1 F(\delta)\,dG(\delta) + \int_0^1 G(\delta)\,dF(\delta) = FG|_0^1 = 0$ this gives

$$\int_0^\infty \sum_{T<|n|<2T} |c_n| \frac{dT}{T} \leqq 8\sqrt{2}\int_0^1 G(\delta)\,dF(\delta).$$

The left side is equal to

$$\sum |c_n| \int_{|n|/2}^{|n|} \frac{dT}{T} = \log 2 \sum |c_n|,$$

furthermore, and so the theorem follows with $A = 8\sqrt{2}/\log 2$.

Bibliography

1. S. Agmon, *Lectures on elliptic boundary value problems*, Van Nostrand Math. Studies, no. 2, Van Nostrand, Princeton, N.J., 1965. MR **31** #2504.

2. L. Bers, F. John and M. Schechter, *Partial differential equations*, Lectures in Appl. Math., vol. 3, Interscience, New York, 1964. MR **29** #346.

3. D. G. de Figueiredo, *The coerciveness problem for forms over vector valued functions*, Comm. Pure Appl. Math. **16** (1963), 63–94. MR **26** #6578.

4. L. Gårding, *Dirichlet's problem for linear elliptic partial differential equations*, Math. Scand. **1** (1953), 55–72. MR **16**, 366.

5. J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris; Academia, Prague, 1967. MR **37** #3168.

6. J. M. Newman, *Coercive inequalities for sectors in the plane*, Comm. Pure Appl. Math. **22** (1969), 825–838.

7. M. Schechter, *Coerciveness of linear partial differential operators for functions satisfying zero Dirichlet-type boundary data*, Comm. Pure Appl. Math. **11** (1958), 153–174. MR **24** #A2747.

8. K. T. Smith, *Inequalities for formally positive integro-differential forms*, Bull. Amer. Math. Soc. **67** (1961), 368–370. MR **26** #462.

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