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A NOTE ON FIXED POINT FREE INVOLUTIONS AND EQUIVARIANT MAPS

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ABSTRACT. The space $P(S^n)$ of all paths ω in S^n with given initial point x and endpoint $-x$ admits an involution $(T\omega)(t) = -\omega(1-t)$. With the standard antipodal involution on S^{n-1} an equivariant map $P(S^n) \to S^{n-1}$ is constructed for $n = 2, 4$, or 8.

A *fixed point free involution* on a space X is a map $T: X \to X$ satisfying $T^2 =$ identity and $Tx \neq x$ for every $x \in X$. Three examples of interest are (i) (S^n, T_1) with $T_1 x = -x$; (ii) $(V(S^n), T_2)$ where $V(X^n) =$ the unit tangent sphere bundle of S^n and $T_2 =$ the antipodal action on each fibre; and (iii) $(P(S^n), T_3)$ where $P(S^n) =$ the space of paths with given initial point $x \in S^n$ and endpoint $-x$, and $(T_3\omega)(t) = -\omega(1-t)$.

For (X, T) a fixed point free involution the *co-index* of X is the least integer n for which there exists an equivariant map $(X, T) \to (S^n, T_1)$. A classical result of Borsuk **[1]** asserts that the co-index of (S^n, T_1) equals n. In **[2]** Conner and Floyd determined the co-index of $(V(S^n), T_2)$ (for all n) and the co-index of $(P(S^n), T_3)$ for all n except $n = 2$, 4, and 8. Their results assert (i) co-index$(V(S^n), T_2) = n$ or $n-1$ according as $n \notin \{1, 3, 7\}$ or $n \in \{1, 3, 7\}$ and (ii) co-index$(P(S^n), T_3) = n$ if $n \notin \{1, 2, 4, 8\}$ and $= n-1$ if $n = 1$. The remaining cases of (ii) are resolved by

PROPOSITION. *For $n = 2$, 4, or 8 there exists an equivariant map* $(P(S^n), T_3) \to (S^{n-1}, T_1)$.

PROOF. For $n = 2$, 4, and 8 there are the Hopf fibrations $S^{n-1} \to S^{2n-1} \xrightarrow{p} S^n$. Here S^{2n-1} is the unit sphere in F^2 ($F =$ complexes for $n = 2$, the quaternions for $n = 4$ and the Cayley numbers for $n = 8$) and the map p assigns to each unit vector the 1-dimensional (over F) subspace it spans. Fix a point $x \in S^n$ and a point $y \in S_x^{n-1} =$ the fibre of p over x. S^{2n-1} is the join of S_x^{n-1} and S_{-x}^{n-1}, where S_{-x}^{n-1} is both the fibre over $-x$ and the unit sphere in the 1-dimensional (over F) subspace orthogonal to the subspace spanned by S_x^{n-1}. Moreover p maps the great circle arc

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$[xz]$ for any $z \in S^{n-1}_{-x}$ 1-1 onto a great circle arc connecting x and $-x$; as z varies over S^{n-1}_{-x} every great circle arc connecting x to $-x$ is realized in the image of p and $p[xz_1] \cup p[xz_2]$ forms a complete great circle passing through x and $-x$ if and only if z_1 and z_2 are antipodal points of S^{n-1}_{-x}. Thus there is a 1-1 map $\varphi: S^n \to S^{2n-1}$ whose image is the union $\bigcup [xz]$ taken over all $z \in S^{n-1}_{-x}$. Now if $E: P(S^n) \times I \to S^n$ denotes the evaluation map, then the composite φE restricts to a map $P(S^n) \times \{1\} \to S^{n-1}_{-x}$ which by the definition of φ sends $(\omega, 1)$ and $(T_3\omega, 1)$ to antipodal points for all ω. This completes the proof.

COROLLARY. *The co-index of* $(P(S^n), T_3)$ *is* $n - 1$ *if* $n \in \{2, 4, 8\}$.

PROOF. As noted in [2] the co-index $(P(S^n)) > n - 2$ for all n, since there is an equivariant embedding $e: (S^{n-1}, T_1) \to (P(S^n), T_3)$ which when composed with any equivariant map $(P(S^n), T_3) \to (S^{n-2}, T_1)$ would provide a contradiction to Borsuk's result quoted above. The Corollary then follows from the Proposition.

REFERENCES

1. K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** (1933), 177–190.

2. P. E. Conner and E. E. Floyd, *Fixed point free involutions and equivariant maps.* II, Trans. Amer. Math. Soc. **105** (1962), 222–228. MR **26** #768.

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