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NAKAYAMA'S LEMMA FOR HALF-EXACT FUNCTORS

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ABSTRACT. We prove an analog of Nakayama's Lemma, in which the finitely generated module is replaced by a half-exact functor from modules to modules. As applications, we obtain simple proofs of Grothendieck's "property of exchange" for a sheaf of modules under base change, and of the "local criterion for flatness."

1. **Nakayama's Lemma.** The form of Nakayama's Lemma we shall start with is:

THEOREM 1.1 [3, §3, PROPOSITION 11]. *Let R be a commutative ring, and N a finitely generated R-module such that for all maximal ideals $\mathfrak{m} \subseteq R$, we have $N = N\mathfrak{m}$ (equivalently, $N \otimes (R/\mathfrak{m}) = \{0\}$). Then $N = \{0\}$.*

PROOF. If N is a nonzero finitely generated R-module, we can find by Zorn's Lemma a maximal proper submodule N_0. Then $N' = N/N_0$ is a simple module (has no proper nonzero submodules). Every simple R-module is isomorphic to one of the form $R/\mathfrak{m}$, for $\mathfrak{m}$ some maximal ideal. Writing N' in this form, we see that $N'\mathfrak{m} = \{0\} \neq N'$, so $N\mathfrak{m} \neq N$.

REMARKS. Taking R local so that there is only one maximal ideal $\mathfrak{m}$, we get one familiar form of Nakayama's Lemma. Another says that $N = \{0\}$ if $N = N\mathfrak{R}(R)$, where $\mathfrak{R}(R) =_{\mathrm{def}} \bigcap \mathfrak{m}$; clearly this also follows from Theorem 1.1. The proof of that theorem can be adapted to non-commutative rings if we replace maximal ideals by right primitive ideals: (2-sided) ideals which are kernels of the action of R on simple right modules, and again one can derive "local" and "Jacobson radical" forms of the lemma.

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2. **R-linear functors.** If R is a commutative ring, we shall denote by $\boldsymbol{M}_R$ the category of all R-modules, and by $\boldsymbol{M}_R^{fg}$ the subcategory of all finitely generated R-modules. These are both additive categories, and furthermore, for any two objects A and B in these categories, Hom (A, B) has a natural structure of R-module. We shall call a functor T between such categories *R-linear* if all the induced maps Hom $(A, B) \to$ Hom $(T(A), T(B))$ are module homomorphisms. Note that if R is Noetherian, $\boldsymbol{M}_R^{fg}$ will be an abelian category.

If N is a finitely generated module over a commutative ring R, the functor $N\otimes : \boldsymbol{M}_R^{fg} \to \boldsymbol{M}_R^{fg}$ will be right-exact, and zero if and only if $N = \{0\}$. Hence the following result generalizes Theorem 1.1 for Noetherian R:

THEOREM 2.1. *Let R be a Noetherian commutative ring, and T an R-linear half-exact functor from $\boldsymbol{M}_R^{fg}$ into itself. If for all maximal ideals $\mathfrak{m} \subseteq R$, $T(R/\mathfrak{m}) = \{0\}$, then $T = 0$.*

(Cf. [**1**, Lemma 6] for an analogous result.)

This will follow as a special case of the next result. If $h: R \to S$ is a homomorphism of commutative rings, we shall call an additive functor $T: \boldsymbol{M}_R^{fg} \to \boldsymbol{M}_S^{fg}$ R-linear (with respect to h) if $T(fr) = T(f)h(r)$ ($f \in$ Map $(\boldsymbol{M}_R^{fg})$, $r \in R$).

THEOREM 2.2. *Let $h: R \to S$ be a homomorphism of commutative rings, suppose R is Noetherian, and let $T: \boldsymbol{M}_R^{fg} \to \boldsymbol{M}_S^{fg}$ be a half-exact R-linear functor. If T annihilates all R-modules of the form $R/h^{-1}(\mathfrak{m})$, for $\mathfrak{m}$ a maximal ideal of S, then $T = 0$.*

PROOF. From the fact that R is Noetherian, it is easy to deduce that any sequence of modules of $\boldsymbol{M}_R^{fg}$: $M_0, M_1, \cdots$ such that each M_{i+1} is a *proper* quotient of a (not necessarily proper) submodule of M_i, must be finite. Hence, given nonzero $M \in \boldsymbol{M}_R^{fg}$, we can assume inductively that for every proper quotient N of a submodule of M, $T(N) = \{0\}$.

Choose a nonzero $x \in M$ maximizing the annihilator ideal $I = \operatorname{Ann} x \subseteq R$. Thus, I will be a prime ideal (an associated ideal of M). We shall show that $T(xR) = \{0\}$. Applying the half-exactness of T to the short exact sequence $\{0\} \to xR \to M \to M/xR \to \{0\}$, we can then conclude that $\{0\} \to T(M) \to \{0\}$ is exact, so $T(M) = \{0\}$ as desired.

If I is of the form $h^{-1}(\mathfrak{m})$ for some maximal ideal $\mathfrak{m} \subseteq S$, $T(xR) \cong T(R/I)$ will be zero by hypothesis. In the contrary case, for every maximal ideal $\mathfrak{m} \subseteq S$ we will have either $I \nsubseteq h^{-1}(\mathfrak{m})$ or $h^{-1}(\mathfrak{m}) \nsubseteq I$. If $I \nsubseteq h^{-1}(\mathfrak{m})$, we note that because I annihilates x, $T(xR)\mathfrak{m} = T(xR)(\mathfrak{m} + h(I)S) = T(xR)$, by maximality of $\mathfrak{m}$. Given $\mathfrak{m}$ such that $h^{-1}(\mathfrak{m}) \nsubseteq I$, choose

$u \in h^{-1}(\mathfrak{m}) - I$. Since I is prime the sequence

$$\{0\} \longrightarrow xR \xrightarrow{u} xR \longrightarrow xR/xuR \longrightarrow \{0\}$$

is exact, whence applying T, so is

$$T(xR) \xrightarrow{u} T(xR) \longrightarrow \{0\}.$$

I.e., $T(xR)u = T(xR)$; hence $T(xR)\mathfrak{m} = T(xR)$. Hence by Theorem 1.1, $T(xR) = \{0\}$, as desired.

COROLLARY 2.3. *Let $h: R \to S$ be a homomorphism of commutative rings, suppose R is Noetherian, and let $T: M_R^{fg} \to M_S^{fg}$ be a half-exact R-linear functor. Let $\mathfrak{p}$ be any prime ideal of S, $S_\mathfrak{p}$ the corresponding localization, and $K_\mathfrak{p}$ its residue field. Then if T annihilates the module $R/h^{-1}(\mathfrak{p})$, T is annihilated by $S_\mathfrak{p}\otimes$; equivalently, by $K_\mathfrak{p}\otimes$.*

PROOF. Since $S_\mathfrak{p}\otimes_S$ is an exact functor, $S_\mathfrak{p}\otimes_S T(\ )$ is a half-exact R-linear functor from M_R^{fg} to $M_{S_\mathfrak{p}}^{fg}$; and by hypothesis, it annihilates the quotient of R by the inverse image of the unique maximal ideal of $S_\mathfrak{p}$. So by Theorem 2.2, $S_\mathfrak{p} \otimes T(\ ) = 0$. Hence so is $(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}) \otimes_S T(\ ) = K_\mathfrak{p} \otimes T$.

The remaining four sections are mutually independent, except that §5 uses §4.

3. **Generalizations and counterexamples.** In Theorem 2.2 we can replace S by a not-necessarily commutative R-algebra, letting $\mathfrak{m}$ run over the primitive ideals of S. To get a still more general statement,[2] consider:

DEFINITION 3.1. An *R-linear* structure on an additive category N will mean a homomorphism of the commutative ring R into the (commutative!) ring of additive natural maps of the identity functor of N into itself.

DEFINITION 3.2. Let N be an additive category with R-linear structure, and $\mathfrak{J}$ a class of ideals of R. N will be said to have the *$\mathfrak{J}$-Nakayama property* if an object A of N is zero when for all $J \in \mathfrak{J}$, either $AJ = A$ or J is properly contained in Ann A; *equivalently* if for all nonzero $A \in N$, $\mathfrak{J}$ contains either Ann A, or some J such that $0 \neq AJ \neq A$.

(If N is not abelian or J not finitely generated we may not be able to define an object AJ in N; but we can always interpret $AJ \neq A$ to mean that there exists a *nonzero* map $f: A \to B$ in N, which is annihilated by all members of J.) Note that an R-linear abelian category N in which every nonzero object can be mapped onto a simple object will be $\mathfrak{J}$-Nakayama if and only if $\mathfrak{J}$ includes the annihilators of all simple objects of N.

[2] The authors are grateful to the referee for correcting an error in an earlier version of this result.

The proof of Theorem 2.2 can be easily adapted to show:

THEOREM 3.3. *Let R be a Noetherian ring, $\mathfrak{J}$ a class of ideals of R, and N a $\mathfrak{J}$-Nakayama R-linear additive category. Then any half-exact R-linear functor $T:\boldsymbol{M}_R^{fg} \to N$ which annihilates R/J for all $J \in \mathfrak{J}$ is zero.*

One can also generalize the domain category $\boldsymbol{M}_R^{fg}$, and the class of test objects $R/\mathfrak{p}$, but we have no generalizations of this sort that are elegant enough to mention here.

Note that in §2 and above, we have had to assume R Noetherian, though in the original Nakayama's Lemma one did not. Let us sketch an example showing that this assumption cannot be dropped. Let k be a field. The set of formal sums $\sum c_\alpha X^\alpha$ with coefficients c_α in k, and non-negative *real* exponents α, such that $\{\alpha \mid c_\alpha \neq 0\}$ is well-ordered, forms a ring of generalized formal power series (cf. [**5**, proof of Theorem VII. 3.8]). This ring R will be local. If we denote by $\mathfrak{m}$ its maximal ideal, one can show that Hom $(\mathfrak{m},\ )$ is an *exact* R-linear functor, and in fact a retraction of the category $\boldsymbol{M}_R^{ef}$ of all R-modules embeddable in finitely generated ones (an abelian category containing $\boldsymbol{M}_R^{fg}$) onto the subcategory $\boldsymbol{M}_R^{fp}$ of finitely presented R-modules (contained in $\boldsymbol{M}_R^{fg}$). This functor annihilates $R/\mathfrak{m}$, but is clearly nonzero!

(Outline of a proof of the above assertions about Hom $(\mathfrak{m},\ )$: Show (i) that every nonzero ideal of R is of one of the forms $x^\alpha R$ or $x^\alpha\mathfrak{m}$, and (ii) that any descending chain of cosets of ideals of R has nonempty intersection. Deduce (ii′), the analog of (ii) for submodules of a free module F of finite rank, and (i′), that every submodule A of a free module F of finite rank can be brought to the form:

$$(e_1x^{\alpha_1}R \oplus \cdots \oplus e_rx^{\alpha_r}R) \oplus (e_{r+1}x^{\alpha_{r+1}}\mathfrak{m} \oplus \cdots \oplus e_sx^{\alpha_s}\mathfrak{m}),$$

where $\{e_1, \cdots, e_t\}$ is a basis for F, and $r \leqq s \leqq t$. Deduce (iii), that for A as above, $\bar{A} =_{\text{def}} \{a \in F \mid a\mathfrak{m} \subseteq A\}$ is free, of finite rank, and (iv), if $M \in \boldsymbol{M}_R^{ef}$, say $M = A/B$, $B \subseteq A \subseteq F$, then Hom $(\mathfrak{m}, M) \simeq \bar{A}/\bar{B} \in \boldsymbol{M}_R^{fp}$.)

For this same ring R, the functor $\text{Tor}^1(k,\ )$ (where we identify k with $R/\mathfrak{m}$) is *left*-exact, on *all of* $\boldsymbol{M}_R$ (because w.gl.dim $R \leqq 1$), takes $\boldsymbol{M}_R^{fg}$ into $\boldsymbol{M}_k^{fg}$, commutes with direct limits, and annihilates $R/\mathfrak{m}$ because $\mathfrak{m}^2 = \mathfrak{m}$, but is nonzero: $\text{Tor}^1(k, R/xR) \simeq k$!

4. **Right-exact functors and tensor products.** Given a homomorphism $h:R \to S$ of commutative rings, what functors $T:\boldsymbol{M}_R^{fg} \to \boldsymbol{M}_S$ can be written in the form $N\otimes_R$ for some S-module N? We note that for any T, there is a unique natural candidate for N, namely, $T(R)$. Furthermore, there will always be a natural map of functors, $t:T(R)\otimes \to T$; namely,

t_M is the element of

$$\mathrm{Hom}_S(T(R) \otimes M, T(M)) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(T(R), T(M)))$$
$$\cong \mathrm{Hom}_R(\mathrm{Hom}_R(R, M), \mathrm{Hom}_S(T(R), T(M)))$$

given by the R-linear functor T, itself.

Let Q designate the cokernel of $t{:}Q(M) = T(M)/t(T(R)\otimes M)$. It is clear that if T has range in $\boldsymbol{M}_S^{fg}$, or $\boldsymbol{M}_S^{ef}$, so does Q, and one finds by diagram-chasing that if T is half-exact (or right-exact), then so is Q. This functor will be our tool for proving:

THEOREM 4.1. *Let $h{:}R \to S$ be a homomorphism of commutative rings, where R is Noetherian, and let $T{:}\boldsymbol{M}_R^{fg} \to \boldsymbol{M}_S^{fg}$ be an R-linear half-exact functor. Then the following conditions are equivalent*:

(i) *T is isomorphic to the functor $T(R)\otimes_R$ (i.e., t is an isomorphism of functors).*

(ii) *The natural map $T(R) \to T(R/h^{-1}(\mathfrak{m}))$ is surjective for all maximal ideals $\mathfrak{m} \subseteq S$.*

(iii) *T is right exact.*

PROOF. (i) $\Rightarrow$ (ii) $\Rightarrow Q(R/h^{-1}(\mathfrak{m})) = \{0\} \Rightarrow Q = 0$ (by Theorem 2.2) $\Rightarrow$ (iii) is clear. Let us prove a slightly more general version of the remaining implication (iii) $\Rightarrow$ (i): delete the assumption that R is Noetherian, and assume T a right-exact R-linear functor from $\boldsymbol{M}_R^{fg}$ to $\boldsymbol{M}_S^{fg}$. We shall show that for M a finitely presented R-module, t_M is an isomorphism. First note that because T is linear it respects finite direct sums, hence t_F is an isomorphism whenever F is free of finite rank. Now let M have a resolution $F_1 \to F_2 \to M \to \{0\}$ with F_1, F_2 free of finite rank. Because T is a right-exact functor defined on a category $\boldsymbol{M}_R^{fg}$ where maps have images, one knows that it preserves exactness of such sequences, so we get $T(R) \otimes F_1 \to T(R) \otimes F_2 \to T(M) \to \{0\}$; so $T(M) = T(R) \otimes M$. (For similar results proved under different assumptions, see [**2**, Theorem II.2.3] and [**6**, III, 7.5.2].)

(In the non-Noetherian example of the preceding section, (i) fails, (ii) holds, "$Q = 0$" fails and (iii) holds! By the above argument however, (i) holds for finitely presented modules.)

If we are interested in functors defined on all of M_R, the following observation is useful in conjunction with Theorem 4.1:

LEMMA 4.2. *Let $h{:}R \to S$ be a homomorphism of commutative rings, and $T{:}\boldsymbol{M}_R \to \boldsymbol{M}_S$ an R-linear functor commuting with direct limits. Then if $t{:}T(R)\otimes \to T$ is an isomorphism for finitely presented R-modules, it is an isomorphism for all R-modules.*

For $T(R)\otimes$ also commutes with direct limits, and every module is a direct limit of finitely presented modules.

One might similarly investigate the relationship between left-exact functors and functors of the form $\mathrm{Hom}_R(N,\ )$ (N an S-module); but this seems a harder question.

5. **Applications to cohomological δ-functors.** Recall that a "cohomological δ-functor $T^\cdot$ from $\boldsymbol{C}$ to $\boldsymbol{C}'$" actually means a sequence $\{T^q\}$ of half-exact additive functors such that whenever $0 \to M' \to M \to M'' \to 0$ is exact in $\boldsymbol{C}$ we get a functorial long exact sequence:

$$\cdots \xrightarrow{\delta} T^q(M') \longrightarrow T^q(M) \longrightarrow T^q(M'') \xrightarrow{\delta} T^{q+1}(M') \longrightarrow \cdots.$$

The present inquiry was motivated by results about such functors which we shall now obtain. The results of the preceding section immediately give:

COROLLARY 5.1. *Let $h: R \to S$ be a homomorphism of commutative rings, where R is Noetherian, and let $T^\cdot: \boldsymbol{M}_R^{fg} \to \boldsymbol{M}_S^{fg}$ be an R-linear cohomological δ-functor. Then for any q, the following conditions are equivalent*:

(1) *For every maximal ideal $\mathfrak{m} \subseteq S$ the natural map $T^q(R) \to T^q(R/h^{-1}(\mathfrak{m}))$ is surjective.*

(2) *For all $M \in \boldsymbol{M}_R^{fg}$ the natural map $T^q(R) \otimes_R M \to T^q(M)$ is an isomorphism.*

(3) *T^q is right-exact.*

(4) *T^{q+1} is left-exact.*

If T extends to a functor $T: \boldsymbol{M}_R \to \boldsymbol{M}_S$ which commutes with direct limits, then the above conditions are also equivalent to:

(2′) *For all $M \in \boldsymbol{M}_R$, $T^q(R) \otimes_R M \to T^q(M)$ is an isomorphism.*

Let us denote the equivalent conditions (1)–(4) of the above corollary by $P(q)$. Since flatness of A is the necessary and sufficient condition for $A\otimes$ to be left-exact, we see:

PROPOSITION 5.2. *Let $h: R \to S$, and $T^\cdot$, be as in the above corollary. Then*

(a) *if $P(q+1)$ holds, $P(q)$ holds if and only if $T^{q+1}(R)$ is flat as an R-module. Hence*:

(b) *If $P(q+1)$ holds, and $T^{q'}(R)$ is flat as an R-module for all $q' \leqq q+1$, $P(q')$ holds for all $q' \leqq q+1$.*

(c) *If for all maximal ideals $\mathfrak{m} \subseteq S$, $T^{q+1}(R/h^{-1}(\mathfrak{m})) = \{0\}$, then $T^{q+1} = 0$ and $P(q)$ holds.*

The extremely useful "property of exchange" of Grothendieck [**6**, III.7.7.5] follows immediately from these results. Let $f: X \to Y$ be a proper morphism of schemes, and $\mathcal{F}$ a sheaf of modules on X, flat over Y. One studies the functor $T^{\cdot}$ on (quasi)coherent $\mathcal{O}_Y$-modules, defined by $T^q(\mathcal{G}) = R^q f_*(\mathcal{F} \otimes f^*\mathcal{G})$. Since all the conditions defining the property $P(q)$ for $T^{\cdot}$ become local statements over Y, one can reduce to the case of Y affine, and get the results of Corollary 5.1 and Proposition 5.2 for this $T^{\cdot}$. ($T^{\cdot}$ will be defined for arbitrary modules and commute with direct limits.)

Some further observations: Given a morphism (natural transformation) $\eta: T \to U$ of R-linear functors T, U: $\boldsymbol{M} \to \boldsymbol{N}$ one can define the kernel and cokernel K, C: $\boldsymbol{M} \to \boldsymbol{N}$ (if $\boldsymbol{N}$ is abelian). In general K and C will not be half-exact if T and U are, which prevents us from applying Theorem 2.2. However, sometimes we can get the necessary half-exactness by special means. We leave the verification of the following example, by diagram-chasing, to the interested reader:

PROPOSITION 5.3. *Let $h: R \to S$ be a homomorphism of commutative Noetherian rings, let $T^{\cdot}$, $U^{\cdot}$: $\boldsymbol{M}_R^{fg} \to \boldsymbol{M}_S^{fg}$ be R-linear cohomological δ-functors, and $\eta^{\cdot}: T^{\cdot} \to U^{\cdot}$ a morphism of functors. If for some q, the maps η^q_M are surjective for all M, and the maps $\eta^{q+1}_{R/h^{-1}(\mathfrak{m})}$ are injective for all maximal ideals $\mathfrak{m} \subseteq S$, then η^{q+1}_M is injective for all M. Similarly, if the maps η^{q+1}_M are injective for all M, and $\eta^q_{R/h^{-1}(\mathfrak{m})}$ surjective for all maximal ideals $\mathfrak{m} \subseteq S$, then η^q_M is surjective for all M.*

6. **Applications: the local criterion for flatness.**

Another result which follows easily from Theorem 2.2 is the local criterion for flatness. In particular, the most useful part, (1) ⇒ (2), follows directly (and does not use the hypothesis that S be Noetherian).

If R is a local ring, with maximal ideal $\mathfrak{m}$, and M an R-module, we shall write $\mathrm{gr}_n\, M$ (or where there is possibility of confusion, $\mathrm{gr}_n^R\, M$) for the $R/\mathfrak{m}$-module $M\mathfrak{m}^n/M\mathfrak{m}^{n+1}$. In particular, $\mathrm{gr}_n\, R = \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

THEOREM 6.1 (cf. [**6**, §5, THEOREM 1]). *Let $h: R \to S$ be a local homomorphism of Noetherian local rings, $\mathfrak{m}$ the maximal ideal of R, and M a finitely generated S-module. Then the following conditions are equivalent:*

(1) $\mathrm{Tor}_1^R(M, R/\mathfrak{m}) = \{0\}$.

(2) *M is flat over R. (I.e.,* $\mathrm{Tor}_1^R(M, \ ) = 0$.)

(3) *$M/M\mathfrak{m}^n$ is flat over $R/\mathfrak{m}^n$ for all n.*

(4) *The canonical surjection $M \otimes \mathrm{gr}_n R \to \mathrm{gr}_n^R M$ is an isomorphism for all n.*

(5) *The maps* $\mathrm{Tor}_1^R(M, R/\mathfrak{m}^{n+1}) \to \mathrm{Tor}_1^R(M, R/\mathfrak{m}^n)$ *are surjective for all $n > 0$.*

PROOF. For (1) ⇒ (2) just apply Theorem 2.2. (2) ⇒ (3) ⇒ (4) ⇒ (5) are easy exercises, and hold for any R-module M. For (5) ⇒ (1), observe that (5) implies that the natural map $\mathbf{lim\,inv}_n\, \mathrm{Tor}_1^R(M, R/\mathfrak{m}^n) \to \mathrm{Tor}_1^R(M, R/\mathfrak{m})$ is surjective. But we claim this inverse limit is zero for any finitely generated S-module M. Indeed, $\mathrm{Tor}_1^R(M, R/\mathfrak{m}^n)$ is a submodule of $M \otimes_R \mathfrak{m}^n$; since the inverse limit is left-exact, it is enough to show that $\mathbf{lim\,inv}\, M \otimes_R \mathfrak{m}^n = \{0\}$. Now since $M \otimes \mathfrak{m}^{n+k} \to M \otimes \mathfrak{m}^n$ factors through $(M \otimes \mathfrak{m}^n)\mathfrak{m}^k$, we see that for each n the image in $M \otimes \mathfrak{m}^n$ of our inverse limit will be contained in $\bigcap_k (M \otimes \mathfrak{m}^n)\mathfrak{m}^k \subseteq \bigcap_k (M \otimes \mathfrak{m}^n)\mathfrak{m}_S^k = \{0\}$, by Krull's Theorem ([**6**, §3, corollary to Proposition 5]). Hence the inverse limit is $\{0\}$.

Note that though conditions (1)–(5) refer only to the R-module structure of M, the proofs of $(1) \Rightarrow (2)$ and $(5) \Rightarrow (1)$ use the structure of finitely generated S-module.

References

1. M. Auslander, *On the dimension of modules and algebras.* III. *Global dimension,* Nagoya Math. J. **9** (1955), 67–77. MR **17**, 579.

2. H. Bass, *Algebraic K-theory,* Benjamin, New York, 1968. MR **40** #2736.

3. N. Bourbaki, *Algèbre commutative.* Chap. 2, Actualités Sci. Indust., no. 1290, Hermann, Paris, 1961. MR **36** #146.

4. ———, *Algèbre commutative.* Chap. 3, Actualités Sci. Indust., no. 1293, Hermann, Paris, 1961. MR **30** #2027.

5. P. M. Cohn, *Universal algebra,* Harper and Row, New York, 1965. MR **31** #224.

6. A. Grothendieck, *Eléments de géométrie algébrique,* Inst. Hautes Études Sci. Publ. Math. No. 11 (1961); ibid. No. 17 (1963). MR **29** #1210; MR **36** #177c.

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