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ON THE FUNCTIONAL EQUATION $\phi(x) = g(x)\phi(\beta(x)) + u(x)$

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ABSTRACT. The linear functional equation of the title is one that has been studied extensively for real or complex x, and for restricted choices of the functions g and β. (See Kuczma **[3]**.) In this paper, we use results of ours **[1]**, combined with an idea due to Diaz and Chu **[2]**, to obtain a powerful existence theorem for continuous solutions of this equation in a generalized form where the domain is an arbitrary compact space, the solutions are vector valued functions, and β is unspecialized, except for continuity.

Let E be a fixed Banach space with norm $|\ |$, and X a compact space. Let $C[X:E]$ be the space of continuous functions f on X to E, normed by $\|f\| = \max_{x\in X} |f(x)|$. Let $C[X]$ be the space of continuous complex valued functions on X, normed in the same way. Let β be any continuous map of X into itself, not necessarily onto or one-to-one. Let $g \in C[X]$. Then, we are interested in conditions under which the general functional equation

$$\phi(x) = g(x)\phi(\beta(x)) + u(x) \tag{1}$$

has a (unique) solution, $\phi \in C[X:E]$ for every $u \in C[X:E]$.

Before stating the main result, several additional concepts must be introduced.

Let $Z = X \times X$ and let K be the compact subset of Z consisting of the diagonal $\Delta = \{\text{all } (x, x) \text{ for } x \in X\}$ and the graph of the map β. Let H be the subspace of $C[Z:E]$ consisting of the functions of the form $f(x, y) = A(x) + B(y)$ where A and B lie in $C[X:E]$. Note that $H|_K$, the restrictions of functions in H to K, is a subspace of $C[K:E]$. Finally, let Γ be the set of fixed points of β, those $x \in X$ for which $\beta(x) = x$.

THEOREM 1. *Assume that g and β obey the following conditions*:

$$|g(x)| < 1 \quad \textit{for all } x \in \Gamma, \tag{2}$$

$$H|_K \quad \textit{is dense in} \quad C[X:E]. \tag{3}$$

Then, (1) *has a* (*unique*) *solution ϕ for every choice of $u \in C[X:E]$.*

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Before giving the proof, we note that (2) cannot be weakened to $|g(x)| \leqq 1$. For, suppose that $g(x_0) = 1$ for some $x_0 \in \Gamma$. Putting $x = x_0$ in (1), we find that $u(x_0)$ must be 0; hence (1) cannot have solutions ϕ for arbitrary functions u.

Our proof will use a standard device. Define a transformation T on $C[X:E]$ by

$$T(f)(x) = g(x)f(\beta(x)) + u(x) \tag{4}$$

and observe that $\|Tf_1 - Tf_2\| \leqq \|g\| \, \|f_1 - f_2\|$. If g is such that $\|g\| < 1$, then T is a contraction, and therefore must have a unique fixed point ϕ obeying $T\phi = \phi$; hence, (1) has a unique solution ϕ.

In [**2**], Diaz and Chu made a simple observation and applied it in a number of illustrations, including a special case of the functional equation (1). The observation is that an operator T has a unique fixed point if this is true of the operator $S^{-1}TS$, where S has S^{-1} for a right inverse.

Following their lead, we make a more complicated choice for S. Let $\psi \in C[X]$ and define a mapping S on $C[X:E]$ by $Sf(x) = f(x) \exp \psi(x)$. Then, it follows that

$$S^{-1}TSf(x) = g_0(x)f(\beta(x)) + u(x)e^{-\psi(x)} \tag{5}$$

where

$$g_0(x) = g(x) \exp \{\psi(\beta(x)) - \psi(x)\}. \tag{6}$$

We now wish to choose ψ so that $\|g_0\| < 1$. This will make $S^{-1}TS$ a contraction operator and thus force T to have a unique fixed point and (1) a solution, for any choice of $u \in C[X:E]$. We note that if $x \in \Gamma$, then $\psi(\beta(x)) - \psi(x) = 0$ so that $g_0(x) = g(x)$. Thus, hypothesis (2) is again forced upon us. Let $\max_{x\in\Gamma} |g(x)| = C$. Then choose C^+, $C < C^+ < 1$, and an open set $\mathcal{O} \supset \Gamma$ such that $|g(x)| < C^+$ for all $x \in \mathcal{O}$. Let $M = \|g\|$, and choose a real valued continuous function v on X such that

$$v(x) \geqq 0 \quad \text{on } X, \tag{7}$$

$$v(x) \geqq \log (M/C^+) \quad \text{off } \mathcal{O}, \tag{8}$$

$$v(x) = 0 \quad \text{on } \Gamma. \tag{9}$$

Suppose that it were possible to choose $\psi \in C[X]$ so that

$$\psi(x) - \psi(\beta(x)) = v(x). \tag{10}$$

[Condition (9) is imposed on v because it is required by (10).] Then, from (6), (7), and (8), it is at once clear that we would have $\|g_0\| \leqq C^+ < 1$ as desired.

In [1], we obtained general theorems dealing with functional equations such as (10). In particular, we proved that a necessary and sufficient condition that (10) have a solution for every function $v \in C[X]$, subject to requirement (9) is that the space $H|_K$ coincides with $C[K]$, and the condition that (10) have arbitrary good approximate solutions is that $H|_K$ is dense. With hypothesis (3), we conclude that for any $\epsilon > 0$, there exists $\psi \in C[X]$ such that $v(x) - \epsilon < \psi(x) - \psi(\beta(x)) < v(x) + \epsilon$ for all $x \in X$. Returning to (6), we now find that $|g_0(x)| \leqq C^+e^\epsilon$ for all $x \in X$, and taking ϵ small enough, $\|g_0\| < 1$.

When $X = [0, 1]$, the results in [1] lead to the following special case.

COROLLARY. *Let β be an increasing continuous function on* $[0, 1]$ *such that the set* $\Gamma = \{$*all* x, $\beta(x) = x\}$ *is finite. Let g be continuous and obey* $|g(x)| < 1$ *on* Γ. *Then, the functional equation* $\phi(x) = g(x)\phi(\beta(x)) + u(x)$ *has a unique continuous solution* $\phi \in C[0, 1]$, *for every choice of the function* $u \in C[0, 1]$.

Further methods being developed to deal with the equation (10) may make it possible to remove the monotonicity condition on β since it is not necessary to be able to solve (10) for all choices of v, but only for some function v obeying (7), (8), (9). We hope to return to this later.

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