C-EMBEDDED SUBSETS OF PRODUCTS

N. NOBLE

Abstract. It is shown that each dense subset of $\mathbb{R}^n$ is $z$-embedded, from which it follows that a dense subset is $C$-embedded if and only if it is $G_δ$-dense. These results extend to, for example, all products of separable metric spaces.

All spaces are assumed to be completely regular Hausdorff; $R$ denotes the real line and $vX$ the Hewitt realcompactification of $X$. Recall that a subset $A$ of $Y$ is $z$-embedded in $Y$ if each zero set of $X$ is the intersection of $X$ with some zero set of $Y$. A subset of $Y$ is $G_δ$-dense in $Y$ if it meets each nonempty $G_δ$-set of $Y$, and the $G_δ$-closure of a subset is the largest subspace in which it is $G_δ$-dense.

Theorem 1. For $X$ a dense subset of $\mathbb{R}^n$, the following conditions are equivalent:

(i) Some superset of $X$ in $\mathbb{R}^n$ is $vX$;
(ii) the $G_δ$-closure of $X$ in $\mathbb{R}^n$ is $vX$;
(iii) the $G_δ$-closure of $X$ in $\mathbb{R}^n$ is realcompact.

Corollary. For $X \subseteq \mathbb{R}^n$, $vX = R$ if and only if $X$ is $G_δ$-dense.

Theorem 1 follows immediately from the fact that each space is $G_δ$-dense in its Hewitt realcompactification but is not $G_δ$-dense in any larger space, a theorem of Hager and Johnson that a $GV$-dense subset is $C$-embedded if and only if it is $z$-embedded [2, Proposition 3] and the following:

Theorem 2. Each dense subspace of $\mathbb{R}^n$ is $z$-embedded.

Proof. Let $X$ be a dense subspace of $\mathbb{R}^n$ and let $Z$ be a zero set in $X$, say $Z = \bigcap U_n$ where each $U_n$ is open and contains the closure in $X$ of $U_{n+1}$. Let $F_n$ be the closure of $U_n$ in $\mathbb{R}^n$; then $F_n \cap X = \text{cl}(U_n)$ so for $F = \bigcap F_n$, $F \cap X = Z$. Thus it suffices to show that $F$ is a zero set.

Since $X$ is dense, each $F_n$ is the closure of its interior, so by [6, Theorem 3] each $F_n$ has the form $π_n^{-1}(H_n)$ where $π_n$ is the projection onto some
countable subproduct and $H_n$ is a closed subspace of that subproduct. It follows that $F$ also has this form, say $F=\pi^{-1}(H)$. But like any closed subset of $R^{n_1}$, $H$ is a zero set. Therefore $F$ is a zero set, as desired.

Notice by the same proof, Theorem 2 holds with $R^n$ replaced by any product space $Y$ satisfying:

(i) Each finite subproduct of $Y$ (and hence $Y$ itself [5, Corollary 1.4]) satisfies the countable chain condition, so the structure theorem for regular closed sets holds [5, Proposition 2.2].

(ii) Each finite subproduct of $Y$ (and hence each countable subproduct of $Y$, by [3, Proposition 2.1]) is perfect, i.e., has each closed subset a $G_\delta$.

(iii) Each countable subproduct of $Y$ has each closed $G_\delta$ a zero set.

In particular, Theorems 1 and 2 hold with $R^n$ replaced by any product of separable metric spaces. Regarding further generalizations, note that if $X$ and $Y$ are pseudocompact subsets of $\beta N$ which contain $N$ and for which $X \times Y$ is not pseudocompact, then $X \times Y$ is $G_\delta$-dense in $\beta N \times \beta N$ but is not $z$-embedded (since if it were it would be $C^*$-embedded which, by Glicksberg’s Theorem, is the case only if $X \times Y$ is pseudocompact). Theorem 1 will be applied in [4] to characterize spaces $Y$ for which $C(Y)$ is realcompact in various standard function space topologies.

REFERENCES


Canary Road, Westlake, Oregon 97493