A COMPACTIFICATION OF LOCALLY COMPACT SPACES

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ABSTRACT. Every locally compact space X has its topology determined by its 1-1 compact images and hence has a compactification \( \xi X \) obtained as the closure of the natural embedding of X in the product of those images, just as the Stone-Čech compactification \( \beta X \) can be obtained by embedding X in a product of intervals. The obvious question is whether \( \xi X = \beta X \). In this paper we prove that \( \xi X = \beta X \) if X either is 0-dimensional or contains an arc, and give an example in which \( \xi X \neq \beta X \).

Preliminaries. All maps are continuous, and all compact and locally compact spaces are Hausdorff. For any space X, let \( \mathcal{K}(X) \) denote the set of all 1-1 maps of X onto a compact space, let \( Y = \prod \{ f(X) \mid f \in \mathcal{K}(X) \} \) and let \( e: X \to Y \) be the evaluation map; if e is a homeomorphism then \( cl_Y e(X) \) is a compactification of X which we denote by \( \xi X \). If X is locally compact then e is necessarily a homeomorphism because for any closed \( F \subseteq X \) there is an \( f \in \mathcal{K}(X) \) such that \( f(F) \) is closed in \( f(X) \); choose \( x \in F \), let \( X' \) be the set X with the topology consisting of all open \( U \subseteq X \) such that either \( x \notin U \) or \( X - U \) is compact, and let \( f: X \to X' \) be the natural map. Note that, by a standard argument, \( \xi X \) is the smallest compactification of X to which every \( f \in \mathcal{K}(X) \) can be extended.

Proposition. Suppose that for any two disjoint zero-sets \( Z_1 \) and \( Z_2 \) of a locally compact space X there is a map f from X into a compact subspace Y of X such that \( f(Z_1) \) and \( f(Z_2) \) have disjoint closures. Then \( \xi X = \beta X \).

Proof. Let \( e: \beta X \to \xi X \) be the Stone extension of the embedding \( e: X \to \xi X \). Since e is a homeomorphism it follows that \( e(\beta X - X) = \xi X - e(X) \); hence we need only show that \( e(p) \neq e(q) \) for any two distinct \( p, q \in \beta X - X \). Let \( A_p \) and \( A_q \) be the free z-ultrafilters on X converging to \( p \) and \( q \), respectively, choose disjoint \( Z_1 \in A_p \) and \( Z_2 \in A_q \), and let f and Y be as hypothesized. Then let \( \tilde{f}: \beta X \to Y \) be the Stone extension of f, let \( Y' = \tilde{f}(\beta X - X) \), let \( X' = (\beta X - X) \cup Y' \), and define \( g: X' \to Y' \) by requiring that...
$g(x)$ be $f(x)$ or $x$ according as $x \in \beta X - X$ or $x \in Y'$. Since $\beta X - X$ and $Y'$ are disjoint closed subsets of $\beta X$, $g$ is continuous. Therefore, since $X'$ is compact and $Y'$ in Hausdorff, $\mathcal{D}' = \{g^{-1}(y) \mid y \in Y'\}$ is a closed, upper semicontinuous decomposition of $X'$, i.e., $\bigcup \{D \in \mathcal{D}' \mid D \cap F \neq \emptyset\}$ is closed for every closed $F \subseteq X'$. As a consequence, if $\mathcal{D} = \mathcal{D}' \cup \{\{x\} \mid x \in \beta X - Y'\}$ then, for any closed $F \subseteq \beta X$,

$$\bigcup \{D \in \mathcal{D} \mid D \cap F \neq \emptyset\} = F \cup \left(\bigcup \{D \in \mathcal{D}' \mid D \cap (F \cap X') \neq \emptyset\}\right)$$

is closed in $\beta X$, so that $\mathcal{D}$ is a closed, upper semicontinuous decomposition of $\beta X$. Thus, if $h$ is the projection of $\beta X$ onto the quotient space determined by $\mathcal{D}$, then $h(\beta X)$ is Hausdorff and hence compact. Let $k = h|X$. Then $k \in \mathcal{K}(X)$ so that there is a map $\tilde{k} : \xi X \rightarrow k(X)$ such that $\tilde{k} \circ e = k$. Now $Z_1 \in \mathcal{A}_p$ so that $p \in \text{cl}_{\beta X} Z_1$ and hence $g(p) = \tilde{f}(p) \in \text{cl} f(Z_1)$. Similarly, $g(q) \in \text{cl} f(Z_2)$. Therefore, since $\text{cl} f(Z_1)$ and $\text{cl} f(Z_2)$ are disjoint, it follows that $g(p) \neq g(q)$ and hence, by the definition of $h$, that $h(p) \neq h(q)$. But $k \circ \tilde{e}$ and $h$ agree on the dense subset $X$ of $\beta X$ so that $k \circ \tilde{e} = h$. Hence $\tilde{e}(p) \neq \tilde{e}(q)$, as required.

**Corollary 1.** If a locally compact space is 0-dimensional in the sense of [3], then $\xi X = \beta X$.

**Proof.** Any two disjoint zero-sets $Z_1$ and $Z_2$ of $X$ are contained in disjoint open sets $U_1$ and $U_2$ whose union is $X$. Choose $x_i \in U_i$, let $Y = \{x_1, x_2\}$, and define $f : X \rightarrow Y$ by requiring $f(x) = x_i$ if $x \in U_i$.

**Corollary 2.** If a locally compact space $X$ contains an arc, then $\xi X = \beta X$.

**Proof.** By assumption, there is a map $g : [0, 1] \rightarrow X$ such that $g(0) \neq g(1)$. For any two disjoint zero-sets $Z_1$ and $Z_2$ of $X$, there is a map $h : X \rightarrow [0, 1]$ such that $h(Z_1) = 0$ and $h(Z_2) = 1$. Let $f = g \circ h$.

**Example.** According to Cook [1] there is a nontrivial, compact, connected space which admits no map into itself other than the identity map and the constant maps. Let $C$ be such a space and let $x$ be the first ordinal with $\text{card} x > \text{card} C$. Note that necessarily card $x$ is uncountable. Now let $x_1$ and $x_2$ be distinct points of $C$, let $p$ be a point not in $C$, and set $H = C \times [0, x] \times [0, x]$ and $K = \{p\} \times [0, x] \times [0, x]$. Then

$$A = \{(x_1) \times \{x\} \times [0, x] \} \cup \{(x_2) \times \{x\} \times [0, x] \}$$

is closed in $H$ and the map $\theta : A \rightarrow K$ defined by requiring $f((x, \beta, \gamma)) = (p, \beta, \gamma)$ is continuous, so that [2, VI.6.1 and VII.3.4] the space $Y = H \cup K$ obtained by “attaching $H$ to $K$ by $\theta$” is compact. Let $\psi : Y' \rightarrow Y$ be the quotient map. Let

$$X' = Y' - ((C \cup \{p\}) \times \{x\} \times \{x\})$$
and let $X = \psi(X')$. Then $Y - X = \psi(Y' - X')$ is closed in $Y$, and hence $X$ is locally compact. Moreover, $X$ is dense in $Y$ so that, in order to show that $\beta X = Y$, it suffices to show that any $f \in C^*(X)$ can be extended to $f \in C^*(Y)$. But $\hat{g} = f \circ \psi(X') \in C^*(X')$ so that, by standard techniques [3, 8L and 9K], one can show that there is a $\beta < \alpha$ such that $g$ is constant on $X' \cap \{x\} \times [\beta, \alpha] \times [\beta, \alpha])$ for all $x \in C \cup \{p\}$. Hence $g$ can be extended to $\hat{g} \in C^*(Y')$ by setting $\hat{g}(\langle x, \alpha, \alpha \rangle) = g(\langle x, \beta, \beta \rangle)$. Moreover, $\hat{g}(\langle x_1, \alpha, \alpha \rangle) = g(\langle x_1, \beta, \beta \rangle) = g(\langle p, \alpha, \beta \rangle) = g(\langle p, \beta, \beta \rangle) = g(\langle p, \alpha, \alpha \rangle)$ and, similarly, $\hat{g}(\langle x_2, \alpha, \alpha \rangle) = g(\langle p, \alpha, \alpha \rangle)$ so that there is an $\hat{f} \in C^*(Y)$ such that $\hat{f} \circ \psi = \hat{g}$, the continuity of $\hat{f}$ following from the fact that $\psi$ is a quotient map. Clearly $\hat{f}$ is the desired extension of $f$ so that $\beta X = Y$ as asserted.

Now consider any $f \in C^*(X)$, let $\tilde{f}: Y \to f(X)$ be the Stone extension of $f$, and let $D = f^{-1}(\tilde{f}(Y - X))$. Then $D$ is a closed subset of $X$ with card $D \leq$ card $(Y - X) = \text{card } C < \text{card } \alpha$, so that $\psi^{-1}(D)$ is a closed subset of $X'$ of cardinality less than card $\alpha$ and hence, by a standard argument, compact. Thus $D$ is compact and hence a continuous image of $Y - X$ under the map $f^{-1} \circ \tilde{f}$. But $Y - X$ is just $C$ with the points $x_1$ and $x_2$ identified, so that $D$ is a continuous image of $C$ under a map which is not 1-1. Therefore, since every component of $X$ is either a singleton or a homeomorph of $C$, it follows that $D$ consists of a single point. Thus $\tilde{f}(Y - X)$ is a singleton so that $f$ can be extended to the one-point compactification of $X$ obtained by identifying $Y - X$ into a point. Hence $\xi X$ is just the one-point compactification of $X$; in particular, $\xi X \neq \beta X$.

Remark. The obvious open problem is to find an internal characterization of the locally compact spaces $X$ for which $\xi X = \beta X$.

REFERENCES


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