REGULAR COMPACTIFICATIONS OF CONVERGENCE SPACES

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Abstract. This note gives a simple characterization for the class of convergence spaces for which regular compactifications exist and shows that each such convergence space has a largest regular compactification.

Introduction. It has been shown by Wyler [5] that for every Hausdorff convergence space $S$ there is a regular (including Hausdorff) compact convergence space $S^*$ and a continuous map $j:S\rightarrow S^*$ with the following property: for every continuous map $f:S\rightarrow T$, where $T$ is regular and compact, there is a unique continuous map $g:S^*\rightarrow T$ such that $f=g\circ j$. Richardson [4] obtained a similar result, but with the following important distinctions: (1) the compactification space $S^*$ is Hausdorff but not necessarily regular (for convergence spaces, Hausdorff plus compact does not imply regular); (2) the map $j$ is a dense embedding. But there is in general no largest Hausdorff compactification, and indeed the number of distinct maximal Hausdorff compactifications can be quite large.

The conclusions of both [4] and [5] suggest that a more satisfactory compactification theory for convergence spaces might result from an investigation of regular compactifications, although it is known (see [2]) that there are regular convergence spaces which cannot be embedded in any compact regular space. What we obtain in this note is a characterization of the class of convergence spaces for which regular compactifications exist, and we show that each such convergence space has a largest regular compactification.

1. For basic information about convergence spaces the reader is asked to refer to [1] and [2]. If there is no possibility of confusion, a convergence space $(S, q)$ will be denoted simply by $S$. A space is regular if it is Hausdorff and has the property: $F$ converges to $x$ implies that the closure of $F$ (denoted $\text{cl } F$) converges to $x$. We shall denote by $\text{cl}_S A$ the closure of a subset $A$ of a convergence space $S$. The pretopological modification $\pi S$ of a convergence space $(S, q)$ is the space $(S, p)$, where $p$ is the finest pretopology...
on $S$ coarser than $q$; the topological modification $\lambda S$ is defined analogously. Recall that $\pi S = \lambda S$ iff the closure operator $cl_S$ is idempotent.

If $A$ is a subset of a convergence space $S$, then the subspace defined by $A$ (also denoted by $A$) is given as follows: A filter $F$ on $A$ $A$-converges to $x$ in $A$ iff the filter on $S$ generated by $F$ $S$-converges to $x$.

**Proposition 1.** The closure operator for a compact regular convergence space is idempotent.

**Proof.** Let $S$ be compact and regular, $A$ a subset of $S$. By the theorem of [4], the identity map from $A$ into $S$ has a continuous extension $f$ to $A^*$, the Stone-Čech compactification of $A$. Since $f(\text{cl}_{A^*} A)=f(A^*) \subseteq \text{cl}_S A$ by continuity of $f$ and $f(A^*)$ is compact and hence $S$-closed, it follows that $\text{cl}_S A$ is closed.

**Proposition 2.** If $A$ is a subspace of $S$, then $\pi A$ is the subspace of $\pi S$ determined by $A$.

**Proposition 3.** If $A$ is a subspace of a compact regular convergence space, then $\pi A$ is Hausdorff and topological.

**Proof.** Let $S$ be a compact regular convergence space containing $A$. By Proposition 2 it suffices to show that $\pi S$ is Hausdorff and topological; that it is topological follows from Proposition 1. To see that $\pi S$ is Hausdorff, let $F$ be an ultrafilter in $\pi S$ which converges to both $x$ and $y$. By compactness $F$ converges in $S$ to some point $z$ and regularity guarantees that $\text{cl}_S F$ also converges in $S$ to $z$. But each neighborhood of $x$ is in $F$, so $x$ is in each member of $\text{cl}_S F$ and hence the Hausdorffness of $S$ implies that $x=z$. Similarly, $y=z$, and so $x=y$.

2. A compactification $(T, f)$ of a convergence space $S$ consists of a compact convergence space $T$ along with a dense embedding $f$ of $S$ into $T$. For a different definition, see §7 of [3].

If $(T, f)$ is a compactification of $S$, then it is a simple matter to verify that $(\pi T, f)$ is a compactification of $\pi S$. From this fact and Proposition 3 we deduce the next result.

**Proposition 4.** If $(T, f)$ is a regular compactification of $S$, then $(\pi T, f)$ is a Hausdorff (topological) compactification of $\pi S$.

**Theorem 1.** A regular convergence space $S$ has a regular compactification iff $\pi S$ is a completely regular topological space and each ultrafilter which is finer than the neighborhood filter at $x$ $S$-converges to $x$ for all $x$ in $S$.

**Proof.** Assume the given conditions. Then $\pi S$ has a topological compactification $(T, f)$. Let $T_1$ be the convergence space consisting of the set $T$ equipped with the finest convergence structure $\tau$ on $T$ which satisfies the following conditions: if $f(S)$ belongs to $\mathcal{G}$, then $\text{cl}_T \mathcal{G} \tau$-converges to $x$.
in \( f(S) \) iff \( f^{-1}(\mathcal{G}) \) \( S \)-converges to \( f^{-1}(x) \); if \( \mathcal{F} \) is an ultrafilter such that \( T - f(S) \) belongs to \( \mathcal{F} \), then \( \text{cl}_T(\mathcal{F}) \) \( r \)-converges to \( x \) in \( f(S) \) iff \( \mathcal{F} \) \( T \)-converges to \( x \); if \( \mathcal{H} \) is an ultrafilter on \( T \), then \( \text{cl}_T \mathcal{H} \) \( r \)-converges to \( y \) in \( T - f(S) \) iff \( \mathcal{H} \) \( T \)-converges to \( y \).

By this construction, it is clear that \( T_1 \) and \( T \) coincide relative to ultrafilter convergence, and so the closure operators for these spaces coincide. The fact that \( T_1 \) is regular can be established with the aid of the following lemma: If \( \mathcal{F} \) is an ultrafilter on \( T - f(S) \) which \( T \)-converges to \( x \) in \( f(S) \), then there is an ultrafilter \( \mathcal{G} \) on \( S \) which \( S \)-converges to \( f^{-1}(x) \) such that \( f^{-1}(\text{cl}_T \mathcal{F}) \supseteq \text{cl}_S \mathcal{G} \). Finally, it is easy to establish that \( f:S\to T_1 \) is a dense embedding.

If \( S \) has a regular compactification \((T,f)\), then \( \pi S \) is a completely regular topological space by Proposition 3. To show that \( S \) has the second property, let \( \mathcal{F} \) be an ultrafilter finer than the \( S \)-neighborhood filter at \( x \). Let \( y \) be the point in \( T \) to which \( f(\mathcal{F}) \) \( T \)-converges. Since \( f(\mathcal{F}) \) is finer than the neighborhood filter for \( f(x) \), \( f(x) \supseteq \text{cl}_T f(\mathcal{F}) \), and so necessarily \( y = f(x) \). Thus \( f^{-1}(f(\mathcal{F})) = \mathcal{F} \) \( S \)-converges to \( x \) since \( f \) is an embedding.

A regular compactification \( T \) of a convergence space \( S \) is a Stone-Šech regular compactification if each continuous function from \( S \) into a compact regular space has a continuous extension to \( T \).

**Theorem 2.** If a convergence space has a regular compactification, then it has a Stone-Šech regular compactification.

**Proof.** Let \( S \) be a convergence space with a regular compactification; let \((T,f)\) be the (topological) Stone-Šech compactification of \( \pi S \), and let \((T_1,f)\) be the regular compactification of \( S \) constructed above. If \( g \) is any continuous function from \( S \) into a compact regular space \( R \), then \( g: \pi S \to \pi R \) has a unique continuous extension \( h:T \to \pi R \) such that \( h \circ f = g \), and it follows easily that \( h:T_1 \to R \) is also continuous.

**References**


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