PSEUDO-ISOTOPIES OF ARCS AND KNOTS

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Abstract. The purpose of this paper is to show that if an arc or simple closed curve contains a nontrivial Wilder arc, then it is not possible to transform a straight line segment onto the arc or a knot onto the simple closed curve. The proof uses the fact that every knot has a unique finite decomposition into prime knots.

A pseudo-isotopy is a homotopy $F_t$, $0 \leq t \leq 1$, such that $F_t$ is a homeomorphism for $t < 1$. An arc $A$ is locally unknotted [4] if for each $x \in A$ there is a neighborhood of $x$ in $A$ that lies on a disk. It is known [5], [6], [7] that if $M$ is a 2-manifold, a locally unknotted arc or a locally unknotted simple closed curve in $E^3$ and if $\varepsilon > 0$, then there is a polyhedral manifold $N \subset E^3$, homeomorphic to $M$, and an $\varepsilon$-pseudo-isotopy $F_\varepsilon$ of $E^3$ that transforms $N$ onto $M$. In addition, $F_0 = 1$, $F_1|N$ is a homeomorphism of $N$ onto $M$ and the set of points whose preimages under $F_1$ are nondegenerate is a zero-dimensional subset of the set of wild points of $M$.

An arc is mildly wild if it is the union of two tame arcs. A Wilder arc [3] is a mildly wild locally peripherally unknotted [4] arc. See Example 1.4 of [1]. The following theorems show that the result above cannot be extended to all arcs and simple closed curves.

Theorem 1. If $A$ is an arc that contains a nontrivial Wilder arc, $I$ is a straight line segment, and $F_t : E^3 \to E^3$ is a pseudo-isotopy, then $F_1(I) \neq A$.

Theorem 2. If $S$ is a simple closed curve that contains a nontrivial Wilder arc, $k$ is a knot, and $F_t : E^3 \to E^3$ is a pseudo-isotopy, then $F_1(k) \neq S$.

However, it is trivial that any arc or simple closed curve can be transformed by a pseudo-isotopy onto a straight line segment or a circle, respectively.

Theorem 1 follows immediately from Theorem 2.

Proof of Theorem 2. Let $C$ be a geometric cone in $E^3$ with vertex $p$ and square base $F_0$. For each integer $i > 0$, let $F_i$ be the intersection of $C$ and the plane parallel to $F_0$ that is half way between $F_{i-1}$ and $p$. Let $C_i$ be

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the closed subset of $C$ between $F_i$ and $F_{i+1}$. Since $S$ contains a nontrivial Wilder arc, we can assume without loss of generality that $S \cap C_0$ is a straight line segment $A_0$ and, for $i > 0$, that $S \cap C_i$ is a polyhedral arc $A_i$ that meets the boundary of $C_i$ only at the endpoints of $A_i$ and the mid-points of $F_i$ and $F_{i+1}$ such that $A_i$ plus an arc on the boundary of $C_i$ is a nontrivial knot.

Suppose there is a knot $k$ in $E^3$ and a pseudo-isotopy $F_t: E^3 \rightarrow E^3$ such that $F_t(k) = S$. For each integer $i \geq 0$, let $N_i$ be a regular neighborhood of $A_i$ in $C_i$ such that $N_i \cap F_{i+1} = N_{i+1} \cap F_{i+1}$.

Let $n$ be a positive integer. There exists a $t < 1$ such that

$$F_t(k) \cap \left( \bigcup_{i=0}^{n+1} C_i \right) \subset \bigcup_{i=0}^{n+1} N_i.$$ We may suppose without loss of generality that $F_t(k) \cap C_0 = A_0$. Then there is a homeomorphism $h: E^3 \rightarrow E^3$ such that

$$h(A_0) = \bigcup_{i=0}^{n} A_i = h(F_t(k)) \cap \left( \bigcup_{i=0}^{n} C_i \right).$$

Since $k$ and $hF_t(k)$ are equivalent, $k$ can be decomposed into at least $n$ knots for every $n$. But every knot has a unique finite decomposition into prime knots [2]. Hence no such pseudo-isotopy exists.

**References**


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