A GENERALIZATION OF MORI'S THEOREM

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Abstract. In this article, we consider a generalization of
Mori's theorem which is: Let $R$ be a Zariski ring; if the completion
of $R$ is a unique factorization domain, then so is $R$.

Mori's theorem states that a Zariski ring $R$ is a unique factorization
domain (UFD) if its completion $\hat{R}$ is a UFD. Validity of the theorem
stems from the facts that $R$ is a Gelfand ring, a filtered ring whose radical
is open, and that $\hat{R}$ is a faithfully flat $R$-module. From this point of view,
a generalization of Mori's theorem is studied in this paper. We prove that
if $R$ is a Gelfand ring whose completion $\hat{R}$ is a flat $R$-module and $R$ is a
pure submodule of $\hat{R}$, in particular, $\hat{R}$ is a faithfully flat $R$-module, then $R$
is a UFD whenever $\hat{R}$ is a UFD. Applying the result, we also consider a
generalization of Nagata-Mori's theorem [5].

In this paper, every ring is assumed to be a commutative ring with
identity. A filtered ring $R$ with a filtration $\{q_n; n=0, 1, 2, 3, \cdots\}$ will be
denoted by $(R, q_n)$, and its completion by $(\hat{R}, \hat{q}_n)$, where $q_0=R$ and $\hat{q}_n$
is the completion of $q_n$. We say that $R$ is hat-flat if $\hat{R}$ is a flat $R$-module
(cf. [1]).

Definition 1. Let $E$ be a module over a ring $R$, and $F$ a submodule of
$E$. $F$ will be called a pure submodule of $E$ if $rE \cap F = rF$ for all $r \in R$.

It is obvious that if the completion $\hat{R}$ of a filtered ring $R$ is a faithfully
flat $R$-module, then $R$ is a pure submodule of $\hat{R}$.

Definition 2. A ring with a linear topology is called a Gelfand ring if
its radical is open (cf. [3, p. 44]).

Note that a hat flat Gelfand ring is necessarily a separated topological
ring by [3, Corollary (5.5)]. Evidently, a filtered ring $(R, q_n)$ is a Gelfand
ring if and only if $q_1 \subseteq \text{rad}(R)$. Thus every Zariski ring $R$ is a hat-flat
Gelfand ring which is a pure submodule of $\hat{R}$ because $\hat{R}$ is a faithfully
flat $R$-module.

The following lemma is easy to verify.

Lemma. Let $B$ be an ideal of a filtered ring $(R, q_n)$. Then $B$ is dense in
$\hat{R}B$ for the $(\hat{q}_nB)$-topology.

Theorem 1. Let $(R, q_n)$ be a Gelfand ring which satisfies the following
conditions: (i) $R$ is hat-flat and (ii) $R$ is a pure submodule of the $R$-module
$\hat{R}$. Then $R$ is a UFD, if $\hat{R}$ is a UFD.

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PROOF. Firstly, $R$ satisfies the ascending chain condition for principal ideals because $\hat{R}$ does and $R$ is a pure submodule of $\hat{R}$. Next, let $a, b \in R$ and $P = Ra \cap Rb$. Since $R$ is hat flat, we have that $\hat{R}P = \hat{R}a \cap \hat{R}b$ by [2, p. 32, Proposition 6]; moreover, $\hat{R}P = \hat{R}c$ for some $c \in \hat{R}P$ as $\hat{R}$ is a UFD. According to the Lemma, there exists a $c_1 \in P$ such that $c \equiv c_1 \mod \hat{q}_1 P$, hence $\hat{R}c = \hat{R}c_1 + \hat{q}_1 P$. Now put $\hat{R}c/\hat{R}c_1 = \hat{R}P/\hat{R}c_1 = E$, then $E$ is a finitely generated $\hat{R}$-module and $\hat{q}_1 E = E$. By [3, Proposition (5.1)], $\hat{R}$ is a Gelfand ring, so that $q_1 \subseteq \text{rad}(\hat{R})$. Applying Nakayama's lemma we have $E = (0)$, that is, $\hat{R}c = \hat{R}c_1$. It follows that $P = Ra \cap Rb = Rc_1$ as $R$ is a pure submodule of $\hat{R}$. Thus the intersection of any two principal ideals of $R$ is principal, which implies that $R$ is a UFD.

COROLLARY 1. Let $R$ be a Gelfand ring such that $\hat{R}$ is a faithfully flat $R$-module. Then $R$ is a UFD, if $\hat{R}$ is a UFD.

The method of the proof of Theorem 1 is the same as the proof of Mori's theorem in [5, p. 2]. We demonstrated that the method still works for the rings with conditions a bit weaker than Zariski rings. Simultaneously, we have shown that Theorem 1, as well as its Corollary 1, is a generalization of Mori's theorem.

COROLLARY 2. Let $(R, q_n)$ be a Noetherian filtered ring satisfying either one of the following two conditions:

(i) $R$ is a hat-flat Gelfand ring,

(ii) the $(q_n)$-topology of $R$ is stronger than its radical topology and $\hat{R}$ is a Noetherian ring. Then $R$ is a UFD, if $\hat{R}$ is a UFD.

PROOF. For case (i), the corollary follows from Corollary 1 to Theorem 1 and [3, Proposition (5.4)].

For case (ii), it is a result of Corollary 1 to Theorem 1 and [4, Proposition (5.4)].

DEFINITION 3. A topological ring is said to be topologically artinian if it is equipped with a linear topology and there exists a fundamental system of neighborhoods of 0 consisting of ideals of finite length.

PROPOSITION. A Gelfand ring $(R, q_n)$ which is topologically artinian is necessarily a quasi-semilocal ring.

PROOF. Since $R$ is a Gelfand ring, $q_1 \subseteq \text{rad}(R)$. Moreover, $R/q_1$ is an artinian ring, hence there exist only a finite number of maximal ideals in $R/q_1$. Now, we can conclude that there exist only a finite number of maximal ideals in $R$, because every maximal ideal of $R$ contains $q_1 \subseteq \text{rad}(R)$.

THEOREM 2. Let $R$ be a hat-flat Gelfand ring which is topologically artinian. Then $R$ is a UFD, if $\hat{R}$ is a UFD.
Proof. The radical of $R$ is open as $R$ is a Gelfand ring, whence every maximal ideal of $R$ is open. Consequently, $\hat{R}$ is a faithfully flat $R$-module due to [1, Proposition 5]. Hence the proposition follows immediately from Corollary 1 to Theorem 1.

In the following Theorem 3 we consider a generalization of Nagata-Mori's theorem [6, p. 56, Theorem 10].

Theorem 3. Let $(R, q_n)$ be a hat-flat ring which is either a Noetherian ring, or a separated topologically artinian ring, satisfying the ascending chain condition (a.c.c.) for principal ideals. If $\hat{R}$ is a UFD and $S=1+q_1$ is generated by prime elements of $R$, then $R$ is also a UFD.

Proof. Suppose that $R$ is a Noetherian hat-flat ring, and put $R'=S^{-1}R$. Then $R'$ is a Gelfand ring and $\hat{R}=\hat{R}$ by Theorem (5.1) and Corollary (5.2) both of [3]. Since $\hat{R}$ is a flat $R$-module, it is a flat $R'$-module by [3, Proposition (5.6)]. Thus $R'$ is a Noetherian hat-flat Gelfand ring. Since $\hat{R}=\hat{R}$ is a UFD, $R'$ is also a UFD by Corollary 2 to Theorem 1. Clearly, $R$ is an integral domain which satisfies the a.c.c. for principal ideals, hence $R$ is a UFD by Nagata's theorem [5, Lemma 1.7]. Next, we assume the other case for $R$. Clearly, $(R', S^{-1}q_n)$ is a Gelfand ring which is topologically artinian, and $R\subseteq R'\subseteq \hat{R}$ because $(\hat{R}, q_n)$ is a Gelfand ring. This implies that $\hat{R}=\hat{R}$ since $(R', S^{-1}q_n)$ is a subspace of $(\hat{R}, q_n)$ due to the fact that $R$ is separated (cf. [6, p. 55, the proof of remark]). Moreover $R'$ is hat-flat by [2, Proposition 6]. Thus $R'$ is a hat-flat Gelfand ring which is topologically artinian. According to Theorem 2, $R'$ is a UFD. Now, that $R$ is a UFD also follows from Nagata's theorem.1

Bibliography


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1 In Nagata's theorem the ring need not be Noetherian if it satisfies the a.c.c. for principal ideals.