AN EXTENSION OF THE NOETHER-DEURING THEOREM

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Abstract. Let $R$ be a commutative semilocal noetherian ring, $\Lambda$ a left noetherian $R$-algebra and $M$, $N$ finitely generated left $\Lambda$-modules such that $\text{End}_\Lambda(M)$ is of finite type over $R$. By $\hat{R}$ we denote the (rad $R$)-adic completion of $R$.

Theorem. $M$ is $\Lambda$-isomorphic to a direct summand of $N$ iff $\hat{R} \otimes_R M$ is $\hat{R} \otimes_R \Lambda$-isomorphic to a direct summand of $\hat{R} \otimes_R N$.

This result is used to prove a generalization of the Noether-Deuring theorem. Let $S$ be a commutative $R$-algebra which is a faithful projective $R$-module of finite type; then $M$ is $\Lambda$-isomorphic to a direct summand of $N$ iff $S \otimes_R M$ is $S \otimes_R \Lambda$-isomorphic to a direct summand of $S \otimes_R N$.

Let $R$ be a semilocal commutative noetherian ring with Jacobson radical $J(R)$ and denote by $\hat{R}$ the $J(R)$-adic completion of $R$; let $S$ be a commutative $R$-algebra such that $S = \hat{R} \otimes_R S$ is a faithful projective $R$-module of finite type. As a generalization of the Noether-Deuring theorem for integral representations we shall prove

Theorem I. Let $\Lambda$ be a left noetherian $R$-algebra, and $M$, $N$ finitely generated left $\Lambda$-modules such that $\text{End}_\Lambda(M)$ is of finite type over $R$. Then $M$ is $\Lambda$-isomorphic to a direct summand of $N$ if and only if $S \otimes_R M$ is $S \otimes_R \Lambda$-isomorphic to a direct summand of $S \otimes_R N$.

It has been pointed out to me by the referee that this is part of a result of A. Grothendieck [8, Proposition 2.5.8.(a)]. A similar statement has also been proven by Białyńicki-Birula [4] using noncommutative Amitsur cohomology and the "théorie de déscente". Our theorem here is the result of an attempt to give a simplified proof of one of the theorems in Białyńicki-Birula's paper.

We shall keep the notation introduced above throughout the paper, and to simplify the notation, we shall write $X \mid Y$ to indicate that $X$ is isomorphic to a direct summand of $Y$. The key role in our proof of
Theorem I is played by

**Theorem II.** Let $\Lambda$ be a left noetherian $R$-algebra, and $M$, $N$ finitely generated left $\Lambda$-modules such that $\text{End}_\Lambda(M)$ is of finite type over $R$. Then

$$M \mid N \text{ as } \Lambda\text{-modules if and only if } \hat{R} \otimes_R M \mid \hat{R} \otimes_R N \text{ as } \hat{R} \otimes_R \Lambda\text{-modules.}$$

We remark that in both theorems the condition that $\text{End}_\Lambda(M)$ is of finite type over $R$ is surely satisfied if $M$ is of finite type over $R$; in fact, $R(a) \rightarrow M \rightarrow 0$ exact, implies $0 \rightarrow \text{End}_\Lambda(M) \rightarrow M(a)$ exact, and so $\text{End}_{\hat{R}}(M)$ is of finite type over $R$, $R$ being noetherian. But $\text{End}_\Lambda(M) \rightarrow \text{End}_\Lambda(M)$ and so $\text{End}_\Lambda(M)$ is of finite type over $R$.

**Proof of Theorem II.** It suffices to prove one direction. So let us assume $\hat{R} \otimes_R M \mid \hat{R} \otimes_R N$ as $\hat{R} \otimes_R \Lambda$-modules. This is equivalent to the existence of a split monomorphism

$$0 \longrightarrow \hat{R} \otimes_R M \longrightarrow \hat{R} \otimes_R N.$$  

Since $\hat{R}$ is a faithfully flat $R$-module (cf. Bourbaki [6, Chapitre III, §3, No 5]) and since $M$ is a finitely generated left module over the left noetherian ring $\Lambda$, we have natural isomorphisms (cf. Auslander-Goldman [1, Lemma 2.4]) for any left $\Lambda$-module $X$.

(1) $\hat{R} \otimes_R \text{Ext}^i_\Lambda(M, X) \cong \text{Ext}^i_{\hat{R} \otimes_R \Lambda}(\hat{R} \otimes_R M, \hat{R} \otimes_R X)$ for $i = 0, 1, \ldots$.

Using this isomorphism for $i = 0$ and identifying both structures, we may write

$$\hat{\sigma} = \sum_{i=1}^n \hat{r}_i \otimes \sigma_i, \quad \hat{r}_i \in \hat{R}, \sigma_i \in \text{Hom}_\Lambda(M, N), 1 \leq i \leq n.$$  

However, $R(J(R) \cong \hat{R} \otimes_R J(R)$, and so we can find elements $r_i \in R$, $1 \leq i \leq n$, such that $\hat{r}_i \otimes \sigma_i \in \hat{R} \otimes_R J(R) = J(\hat{R})$. To prove Theorem II, we have to establish the existence of a split monomorphism $0 \rightarrow M \rightarrow N$. We claim that $\sigma = \sum_{i=1}^n r_i \sigma_i \in \text{Hom}_\Lambda(M, N)$ has the desired properties. Since $\hat{R}$ is a faithfully flat $R$-module, it suffices to show that $1_{\hat{R} \otimes_R} \sigma$ is a split monomorphism. In fact, assuming that $1_{\hat{R} \otimes_R} \sigma$ is a split monomorphism, $\sigma$ must be monic, and it remains to show that the sequence

$$E : 0 \rightarrow M \rightarrow N \rightarrow N/\text{Im } \sigma \rightarrow 0$$  

is split exact. We consider the $R$-submodule $X$ of $\text{Ext}^1_\Lambda(N/\text{Im } \sigma, M)$ generated by the class of $E$. Because of the isomorphism (1) for $i = 1$ we have $\hat{R} \otimes_R X = 0$ and so $X$ must be zero; i.e., $E$ is split exact. It remains to show that $1_{\hat{R} \otimes_R} \sigma$ is a split monomorphism. Since $\sigma$ was a split monomorphism to start with, there exists $\tilde{\sigma} \in \text{Hom}_{\hat{R} \otimes_R \Lambda}(\hat{R} \otimes_R N, \hat{R} \otimes_R M)$ such
that $\delta \hat{\varphi} = 1_{\hat{R} \otimes R M}$. But then
\[
(1_B \otimes \sigma) \hat{\varphi} - 1_{\hat{R} \otimes R M} = (1_B \otimes \sigma - \delta) \hat{\varphi} = \left[ \sum_{i=1}^{n} (r_i - \hat{r}_i) \otimes \sigma_i \right] \hat{\varphi} \in J(\hat{R}) \text{End}_{\hat{R} \otimes R \Lambda}(\hat{R} \otimes R M).
\]

But $\text{End}_{\hat{R} \otimes R \Lambda}(\hat{R} \otimes R M)$ is of finite type over $\hat{R}$ and so $J(\hat{R}) \text{End}_{\hat{R} \otimes R \Lambda}(\hat{R} \otimes R M)$ is contained in the Jacobson radical of $\text{End}_{\hat{R} \otimes R \Lambda}(\hat{R} \otimes R M)$ (cf. Bourbaki \[5, \text{Chapitre VIII, §6, N° 3, Théorème 2}\]) and so $(1_B \otimes \sigma) \hat{\varphi}$ is a unit in $\text{End}_{\hat{R} \otimes R \Lambda}(\hat{R} \otimes R M)$; i.e., $1_B \otimes \sigma$ is a split monomorphism and so $M \cong \bar{N}$. Q.E.D.

**Corollary 1.** Let $\Lambda$ be an $R$-algebra and $M$ a finitely presented left $\Lambda$-module such that $\text{End}_{\Lambda}(M)$ is of finite type over $R$ and $N$ a left $\Lambda$-module. Then $M \cong \bar{N}$ if and only if $\hat{R} \otimes R M \cong \hat{R} \otimes R N$.

The proof is similar to the one of Theorem II; however, we do not need the assumption that $\Lambda$ is left noetherian, since (1) is valid for $i=0$ also for a finitely presented $\Lambda$-module $M$.

**Remark.** Under the assumptions of Corollary 1, the Krull-Schmidt theorem is valid for the indecomposable direct summands of $\hat{R} \otimes R M$. For this it suffices to know, that for each indecomposable direct summand $X$ of $\hat{R} \otimes R M$, the ring $\text{End}_{\hat{R} \otimes R \Lambda}(X)$ is complete with respect to the topology induced by $J(\hat{R}) \text{End}_{\hat{R} \otimes R \Lambda}(X)$ (cf. Bass \[3, \text{Chapter III, Proposition 2.10}\]); but this is clear since $\text{End}_{\hat{R} \otimes R \Lambda}(X)$ is of finite type over $\hat{R}$ (cf. Bourbaki \[6, \text{Chapitre III, §3, N° 4, Théorème 3}\]).

**Corollary 2.** Under the assumptions of Corollary 1, let $X$ be a finitely presented left $\Lambda$-module such that $\text{End}_{\Lambda}(M \oplus X)$ is of finite type over $R$. Then $M \oplus X \cong N \oplus X$ implies $M \cong N$.

**Proof.** This is an immediate consequence of Corollary 1 and the Remark.

**Corollary 3.** Under the assumptions of Corollary 1, $M^{(n)} \cong N^{(n)}$ implies $M \cong N$.

**Proof.** This follows from Corollary 1 and the Remark.

**Corollary 4.** Let $\mu_{\Lambda}(X)$ denote the minimal number of $\Lambda$-generators of the left $\Lambda$-module $X$, where $\Lambda$ is an $R$-algebra. Assume that $M$ is a finitely presented left $\Lambda$-module which is of finite type over $R$. Then $\mu_{\Lambda}(M) \leq n$ if and only if $\mu_{\hat{R} \otimes R \Lambda}(\hat{R} \otimes R M) \leq n$.

**Proof.** Since the tensor product is right exact, it suffices to prove one direction. Let $\mu_{\hat{R} \otimes R \Lambda}(\hat{R} \otimes R M) \leq n$. Then we have an epimorphism
\[
(\hat{R} \otimes R \Lambda)^{(n)} \xrightarrow{\delta} \hat{R} \otimes R M \longrightarrow 0.
\]
As in the proof of Theorem II, we construct \( \sigma \in \text{Hom}_A(\Lambda^{(n)}, M) \) such that 
\[ (1_R \otimes \sigma) - \delta \in J(\bar{R})\text{Hom}_{R \otimes R A}(\bar{R} \otimes_R \Lambda^{(n)}, \bar{R} \otimes_R M). \]
But then \( \text{Im}(1_R \otimes \sigma) + J(\bar{R})(\bar{R} \otimes_R M) = \bar{R} \otimes_R M \) and Nakayama's Lemma shows that \( 1_R \otimes \sigma \) must be an epimorphism. However, \( \bar{R} \otimes_R \) — is faithfully flat, and so \( \varphi \) is an epimorphism.

Finally we turn to the proof of Theorem I.

It suffices to prove one direction. Let \( S \otimes_R M | S \otimes_R N \). Then \( S \otimes_R M | S \otimes_R N \) as \( S \otimes_R \Lambda \)-modules. However, \( J(R) = \cap_{i=1}^s m_i \), where \( \{m_i\}_{1 \leq i \leq s} \) are the maximal ideals of \( R \). Then \( \bar{R} = \prod_{i=1}^s \bar{R}_i \) is the product of complete local rings. Since we have assumed \( S \otimes_R \) to be a faithful projective \( \bar{R} \)-module of finite type, we have
\[
S \cong \bigoplus_{i=1}^s \bar{R}_i^{(n_i)}, \quad n_i > 0, \quad 1 \leq i \leq s.
\]
Thus \( S \otimes_R M | S \otimes_R N \) as \( S \otimes_R \Lambda \)-modules implies
\[
(\bar{R}_i \otimes_R M)^{(n_i)} | (\bar{R}_i \otimes_R N)^{(n_i)}, \quad 1 \leq i \leq s
\]
as \( \bar{R}_i \otimes_R \Lambda \)-modules. Now the Krull-Schmidt theorem shows
\[
\bar{R}_i \otimes_R M | \bar{R}_i \otimes_R N, \quad 1 \leq i \leq s,
\]
and so \( \bar{R} \otimes_R M | \bar{R} \otimes_R N \). An application of Theorem II gives the desired result: \( M | N \). Q.E.D.

References


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