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ON ASYMPTOTIC BEHAVIOR OF PERTURBED NONLINEAR SYSTEMS

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ABSTRACT. A version of the variation of constants formula for nonlinear systems is used to study the comparative asymptotic behavior of the systems $x'=f(t, x)$ and $y'=f(t, y)+g(t, y)$.

1. **Introduction.** Marlin and Struble [6] showed the asymptotic behavior of the solutions of a nonlinear system determines the asymptotic behavior of perturbations of the system. This paper is a continuation of this problem. We place hypotheses on the basic nonlinear system and its perturbation analogous to those placed on a linear system and its perturbation by Brauer and Wong [2] and Hallam and Heidel [4]. The basic tools of the investigation are a generalized version of the variation of constants formula, a much used comparison principle and fixed point theorems. The proofs are similar to those given in [2] and [4] for perturbations of a linear system.

Let α be a real number, let Ω be a region in R^n, let f and g be continuous functions from $[\alpha, \infty)\times\Omega$ into R^n such that $f_x(t, x)$ exists and is continuous on $[\alpha, \infty)\times\Omega$ and consider the differential equations

$$x' = f(t, x), \tag{1}$$

$$y' = f(t, y) + g(t, y). \tag{2}$$

Let $x(t, \tau, \gamma)$ denote the solution of (1) passing through γ at $t=\tau$, let Ω_1 be open and such that $\bar{\Omega}_1\subset\Omega$ and assume that for γ in $\bar{\Omega}_1$, $\alpha\leqq\tau$, the function $x(t, \tau, \gamma)$ exists for $\alpha\leqq t$. It is known [3] the derivative matrix

$$\frac{\partial x}{\partial \gamma}(t, \tau, \gamma) = \Phi(t, \tau, \gamma)$$

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exists, satisfies the variational equation

$$z' = f_x(t, x(t, \tau, \gamma))z,$$

$\Phi(\tau, \tau, \gamma) = I$, and

$$\frac{\partial x}{\partial \tau}(t, \tau, \gamma) = -\Phi(t, \tau, \gamma)f(\tau, \gamma).$$

Suppose $w(t, \lambda)$ is a continuous nonnegative function on $[\alpha, \infty) \times R^+$, is nondecreasing in λ, and for some $0 < k$ there is a unique solution of

$$r' = w(t, r), \qquad r(\alpha) = k. \tag{3}$$

Further we assume $r(t)$, the solution of (3), exists for $t \geqq \alpha$ and $\lim_{t \to \infty} r(t) = r_\infty$. Notice that if $0 \leqq \lambda < r_\infty$ we have $\int_\alpha^\infty w(t, \lambda)\, dt < \infty$. This follows since $\int_\alpha^\infty w(s, r(s))\, ds$ exists, so for t_1 such that $\lambda \leqq r(t)$ for $t \geqq t_1$,

$$0 \leqq \int_{t_1}^\infty w(s, \lambda)\, ds \leqq \int_{t_1}^\infty w(s, r(s))\, ds < \infty.$$

2. **Main results.** Theorem 1 below establishes that corresponding to some solutions of (2) which exist for $t \geqq \alpha$ there is a solution of (1) which is asymptotically similar and Theorem 2 investigates the converse problem.

THEOREM 1. *Let $D(t)$ be a continuous nonsingular $n \times n$ matrix for $t \geqq \alpha$ and $\Omega_2 \subset \Omega_1$ be such that*

(a) *$t \geqq \alpha$, $|D(t)\gamma| \leqq r_\infty$ implies γ is in Ω_1,*

(b) *$|D(t)\Phi(t, \tau, \gamma)g(t, \gamma)| \leqq w(\tau, |D(\tau)\gamma|)$, for t, τ in $[\alpha, \infty)$, γ in Ω_1,*

(c) *$|D(t)x(t, \alpha, \gamma)| \leqq k$, for $t \geqq \alpha$, γ in Ω_2.*

Then for γ in Ω_2 there is a solution $y(t)$, $\alpha \leqq t$, of (2) passing through γ at $t = \alpha$; and for each such solution there is a corresponding solution x^*, $\alpha \leqq t$, of (1) such that

$$\lim_{t \to \infty} D(t)[y(t) - x^*(t)] = 0. \tag{4}$$

PROOF. It is known [1] that for γ in Ω_2 the solutions of (2) passing through γ at $t = \alpha$ satisfy

$$y(t) = x(t, \alpha, \gamma) + \int_\alpha^t \Phi(t, s, y(s))g(s, y(s))\, ds \tag{5}$$

for all t for which $y(t)$ is in Ω_1. We have

$$|D(t)y(t)| \leqq k + \int_\alpha^t w(s, |D(s)y(s)|)\, ds;$$

thus by standard inequality theorem [5, p. 29]

$$|D(t)y(t)| \leqq r(t) \leqq r_\infty.$$

By (a) in the hypothesis $y(t)$ is in Ω_1; therefore y is defined and satisfies (5) for all $t \geqq \alpha$. Further for $\varepsilon > 0$ there is a T such that when $T \leqq t_1 \leqq t_2$ we have $0 < r(t_2) - r(t_1) < \varepsilon$. Consequently

$$\left| \int_{t_1}^{t_2} D(t)\Phi(t, s, y(s))g(s, y(s))\, ds \right| \leqq \int_{t_1}^{t_2} w(s, r(s))\, ds = r(t_2) - r(t_1) < \varepsilon;$$

therefore $\int_\alpha^\infty \Phi(t, s, y(s))g(s, y(s))\, ds$ exists uniformly for t in compact intervals. Now

$$D(t)y(t) = D(t)\left[x(t, \alpha, \gamma) + \int_\alpha^\infty \Phi(t, s, y(s))g(s, y(s))\, ds\right] - D(t)\int_t^\infty \Phi(t, s, y(s))g(s, y(s))\, ds;$$

thus

$$\left| D(t)\left[y(t) - x(t, \alpha, \gamma) - \int_\alpha^\infty \Phi(t, s, y(s))g(s, y(s))\, ds\right] \right| \leqq \int_t^\infty w(s, r(s))\, ds = r_\infty - r(t) \to 0$$

as $t \to \infty$. It remains to show

$$x^*(t) = x(t, \alpha, \gamma) + \int_\alpha^\infty \Phi(t, s, y(s))g(s, y(s))\, ds$$

is a solution of (1). We have

$$x(t, T, y(T)) + \int_\alpha^\infty \Phi(t, s, y(s))g(s, y(s))\, ds - x^*(t) = \int_\alpha^T \frac{d}{ds} x(t, s, y(s))\, ds = \int_\alpha^T \Phi(t, s, y(s))g(s, y(s))\, ds,$$

which implies $\lim_{T\to\infty} x(t, T, y(T)) = x^*(t)$ uniformly for t in compact intervals. Also

$$f(t, x^*(t)) - f(t, x(t, \alpha, \gamma)) = \lim_{T\to\infty} \int_\alpha^T \frac{d}{ds}[f(t, x(t, s, y(s)))]\, ds = \lim_{T\to\infty} \int_\alpha^T f_x(t, x(t, s, y(s)))\Phi(t, s, y(s))g(s, y(s))\, ds$$

uniformly for t in compact intervals. Therefore

$$x^{*\prime}(t) = f(t, x(t, \alpha, \gamma)) + \int_\alpha^\infty f_x(t, x(t, s, y(s)))\Phi(t, s, y(s))g(s, y(s))\, ds = f(t, x^*(t)).$$

Theorem 2 below investigates the converse problem to that considered in Theorem 1. In the case f is linear in x this problem [4] amounts to showing there is a solution of the integral equation

$$y(t) = x(t) - \int_t^\infty \Phi(t, s, y(s))g(s, y(s))\, ds.$$

This can be done using the Schauder fixed point theorem for general f; however if f is nonlinear it is not readily apparent that such a solution satisfies (2).

THEOREM 2. *Let the hypothesis of Theorem* 1 *hold. Then for any solution* x *of* (1) *which exists for* $t \geqq \alpha$ *and such that* $x(\alpha)$ *is in* Ω_2, *there is a* $t_1 \geqq \alpha$ *and a solution* y *of* (2) *for* $t \geqq t_1$ *such that* $\lim_{t\to\infty} D(t)[y(t)-x(t)]=0$.

PROOF. We begin by constructing a sequence of solutions of (2) which is uniformly convergent on compact subintervals. Let $0<\sigma<r_\infty-k$, choose $t_1 \geqq \alpha$ so large that $\int_{t_1}^\infty w(s, \sigma+k)\, ds<\sigma$; and for any integer $n \geqq t_1$ let $\mathscr{F}_n$ be the set of continuous functions, v, from $[t_1, n]$ into R^n satisfying

$$|v(t)| \leqq |D(t)x(t)| + \sigma \quad \text{for } t_1 \leqq t \leqq n.$$

First we show that (2) has a solution v_n in $\mathscr{F}_n$. Define an operator S on $\mathscr{F}_n$ by

$$Sv(t) = D(t)x(t) + \int_n^t D(t)\Phi(t, s, D^{-1}(s)v(s))g(s, D^{-1}(s)v(s))\, ds$$

for $t_1 \leqq t \leqq n$. We observe

$$|Sv(t) - D(t)x(t)| \leqq \int_t^n w(s, |v(s)|)\, ds \leqq \int_t^n w(s, \sigma + k)\, ds \leqq \sigma;$$

thus S maps $\mathscr{F}_n$ into itself. It is easy to see the hypothesis of the Schauder fixed point theorem holds; thus there is a fixed point of S in $\mathscr{F}_n$ which we denote by v_n. The function y_n given by $y_n(t)=D^{-1}(t)v_n(t)$ satisfies

$$y_n'(t) = f(t, x(t)) + g(t, y_n(t)) + \int_n^t f_x(t, x(t, s, y_n(s)))\Phi(t, s, y_n(s))g(s, y_n(s))\, ds;$$

and since

$$f(t, x(t, t, y_n(t))) - f(t, x(t)) = \int_n^t \frac{d}{ds} f(t, x(t, s, y_n(s)))\, ds$$

$$= \int_n^t f_x(t, s, y_n(s))\Phi(t, s, y_n(s))[y_n'(s) - f(s, y_n(s))\, ds],$$

we have

$$\omega(t) = -\int_n^t f_x(t, x(t, s, y_n(s)))\Phi(t, s, y_n(s))\omega(s)\, ds$$

where $\omega(t)=y_n'(t)-f(t, y_n(t))-g(t, y_n(t))$. But this implies $\omega=0$ thus y_n is a solution of (2) on $[t_1, n]$.

Let N be an integer larger than t_1 and consider the sequence v_n, $n=N, N+1, \cdots$, of fixed points obtained above. Clearly $|v_n(t)|\leqq k+\sigma$ for $t_1\leqq t\leqq N$ and the sequence $\{v_n\}_N^\infty$ is equicontinuous on this interval.

By Ascoli's theorem there is a subsequence $\{v_{n1}\}$ of the v_n's converging uniformly on $[t_1, N]$. Similarly the sequence $\{v_{n1}\}$ is defined on $[t_1, N+1]$ for $n1\geqq N+1$ and is equicontinuous on $[t_1, N+1]$ so there is a subsequence of the v_{n1}'s say $\{v_{n2}\}$ converging uniformly on this interval. Clearly on the interval $[t_1, N]$ both subsequences converge to the same limit. Proceeding inductively we define a function v on $[t_1, \infty)$ and a chain of subsequences $\{v_{nk}\}$ such that $\{v_{nk}\}$ converges uniformly to v on $[t_1, N+k]$. The sequence $\{\bar{v}_n\}_1^\infty$, where $\bar{v}_n=v_{nn}$, then converges to v uniformly on compact subintervals of $[t_1, \infty)$.

By using hypothesis (b) it is easy to see that

$$\int_t^\infty D(t)\Phi(t, s, D^{-1}(s)v(s))g(s, D^{-1}(s)v(s))\, ds$$

exists and that

$$\lim_{n\to\infty}\int_t^n D(t)\Phi(t, s, D^{-1}(s)\bar{v}_n(s))g(s, D^{-1}(s)\bar{v}_n(s))\, ds$$
$$=\int_t^\infty D(t)\Phi(t, s, D^{-1}(s)v(s))g(s, D^{-1}(s)v(s))\, ds.$$

The functions $\bar{y}_n(t)=D^{-1}(t)\bar{v}_n(t)$ are solutions of (2); consequently $y(t)=D^{-1}(t)v(t)$ is also a solution and we note

$$D(t)[y(t)-x(t)] = -\int_t^\infty D(t)\Phi(t, s, y(s))g(s, y(s))\, ds$$

which vanishes at $t\to\infty$.

Marlin and Struble [**6**] consider the case $D(t)=I$ and Ω_1 is bounded and convex. By arguments used in the proof of Theorem 2 above it is possible to improve the corresponding theorem in [**6**].

THEOREM 3. *Assume Ω_1 is bounded and x is a solution of* (1) *with values in Ω_1 for $t\geqq\alpha$ and without limit points on the boundary of Ω_1. Further assume that whenever z is a continuous function from* $[\alpha, \infty)$ *into Ω_1 and $\alpha\leqq t\leqq T$ then*

$$\int_T^\infty |\Phi(t, s, z(s))g(s, z(s))|\, ds \leqq J(T)$$

where $J(T)\to 0$ as $T\to\infty$. Then there is a $t_1\geqq\alpha$ and a solution y of (2) *for $t\geqq t_1$ such that* $\lim_{t\to\infty}[y(t)-x(t)]=0$.

3. **An example.** The following example is similar to an example given by Brauer and Wong in [**2**] and shows that the asymptotic behavior of unbounded solutions can be compared. This feature is not present in the theorems of Marlin and Struble [**6**]. Let h be a continuous real valued function defined on $[0, \infty)\times R^2$ which satisfies

$$|h(t, u, \dot{u})| \leqq h_0(t)\,|u|^m + h_1(t)\,|\dot{u}|^n,$$

where h_0 and h_1 are continuous functions and m and n are positive numbers with $p=\max\{m, n\}>1$. If

$$\int_1^\infty \tau^{m+1}h_0(\tau)\,d\tau < \infty, \qquad \int_1^\infty h_1(\tau)\,d\tau < \infty,$$

α is large enough and $\max\{|\gamma_1|, |\gamma_2|\}\leqq\frac{1}{2}$ there is a solution u of

$$\ddot{u} + e^{-t}\dot{u}^2 = h(t, u, \dot{u}), \qquad u(\alpha) = \gamma_1, \qquad \dot{u}(\alpha) = \gamma_2,$$

u exists for $t\geqq\alpha$ and there is a constant δ such that $u(t)=\delta t+o(t)$ as $t\to\infty$. Here we used

$$D(t) = \begin{bmatrix} \frac{1}{3}t & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Omega_1 = \{\gamma \text{ in } R^2;\ |\gamma_2| < 2\},$$

$$\Omega_2 = \{\gamma \text{ in } R^2\text{: } \max\{|\gamma_1|, |\gamma_2| \leqq \tfrac{1}{2}\}\},$$

$$w(t, r) = \lambda(t)\max\{r^p, 1\}$$

where λ is constructed as in [**2**].

References

1. Fred Brauer, *Perturbations of nonlinear systems of differential equations*, J. Math. Anal. Appl. **14** (1966), 198–206. MR **33** #359.

2. Fred Brauer and J. S. W. Wong, *On asymptotic behavior of perturbed linear systems*, J. Differential Equations **6** (1969), 142–153. MR **39** #570.

3. W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath, Boston, Mass., 1965. MR **32** #7875.

4. T. G. Hallam and J. W. Heidel, *The asymptotic manifolds of a perturbed linear system of differential equations*, Trans. Amer. Math. Soc. **149** (1970), 233–241. MR **41** #2136.

5. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964. MR **30** #1270.

6. J. A. Marlin and R. A. Struble, *Asymptotic equivalence of nonlinear systems*, J. Differential Equations **6** (1969), 578–596. MR **40** #5985.

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