THE WALL INVARIANT OF CERTAIN $S^1$ BUNDLES

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Abstract. Let $p: E \to B$ be a principal $S^1$ bundle with $B$ dominated by a finite complex. Then it is easy to show that $E$ is also dominated by a finite complex. In this paper we show, under suitable additional hypotheses, that in fact $E$ has the homotopy type of a finite complex. The proof is carried out by computing Wall's finiteness obstruction for $E$.

Let $X$ be a (possibly infinite) CW complex which is dominated by a finite complex and let $\Lambda = Z(\pi_1(X))$. In $[8]$ Wall defines an obstruction $\sigma(X) \in \bar{K}_0(\Lambda)$ to finding a finite complex with the same homotopy type as $X$.

If $F \to E \to B$ is a fiber space with total space, base, and fiber dominated by finite CW complexes, it is natural to ask whether $\sigma(E)$ can be computed in terms of $\sigma(F)$, $\sigma(B)$, and invariants of the fiber space. Since $\sigma(F)$, $\sigma(E)$, and $\sigma(B)$ are elements of $\bar{K}_0Z(\pi_1(F))$, $\bar{K}_0Z(\pi_1(E))$, and $\bar{K}_0Z(\pi_1(B))$ respectively, and $\cdots \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \cdots$ is exact, the solution of the general problem involves an extension problem. In an effort to obtain a better understanding of the relationship between these problems, we will prove the

Main Theorem. Let $p: E \to B$ be a principal $S^1$ bundle such that

(i) $B$ is a CW complex dominated by a finite complex;
(ii) $j_#: \pi_1(S^1) \to \pi_1(E)$ is a monomorphism; and
(iii) $\pi_1(E)$ is abelian.

Then $E$ is dominated by a finite complex and $\sigma(E) = 0$.

The universal covering space $\tilde{E}$ of $E$ and the action of $\pi_1(E)$ on $\tilde{E}$ are determined in §1. This information is then used in §2 to prove the theorem.

The problem considered here was first raised by Lal in [3]. Unfortunately, the main theorems of that paper (Theorems 2 and 3) do not hold in the generality Lal claims, but appear to require additional hypotheses. A forthcoming paper by the author [0] contains a counterexample to Lal’s Theorem 3 as well as a modified version of Lal’s Theorem 3.
1. $\tilde{E}$ and the $\pi_1(E)$ action on $\tilde{E}$. It is the object of this section to prove the

**Theorem 1.1.** Let $p:E\to B$ be a principal $S^1$ bundle satisfying the hypothesis of the Main Theorem. Then $\tilde{E}$ is homeomorphic to $\tilde{B} \times \mathbb{R}$. Furthermore there is a homomorphism $\tau: \pi_1(E) \to \mathbb{Q}$ with image an infinite cyclic subgroup of $\mathbb{Q}$ and a choice of homeomorphism $h: \tilde{B} \times \mathbb{R} \to \tilde{E}$ such that $h$ is equivariant with respect to the $\pi_1(E)$ action on $\tilde{B} \times \mathbb{R}$ defined by $\alpha(b, t) = (p#\alpha b, t + \tau(\alpha))$ for any $\alpha \in \pi_1(E)$ and the covering transformation action of $\pi_1(E)$ on $\tilde{E}$.

The $\mathbb{Q}$ above denotes the additive group of rational numbers and the juxtaposition $p#\alpha b$ denotes the covering transformation action of $\pi_1(B)$ on $\tilde{B}$.

The basic idea of the proof of 1.1 is to apply the

**Lemma 1.2.** Let $p':L\to K$ be a principal $S^1$ bundle with base space a $K(\pi_1(K), 1)$. The isomorphism class of the extension

$$0 \to \pi_1(S^1) \xrightarrow{\partial'} \pi_1(L) \xrightarrow{p'} \pi_1(K) \to 1$$

is a complete invariant of the bundle.

**Proof.** It is well known that the Chern class $c_1 \in H^2(K; \mathbb{Z})$ is a complete invariant of the bundle $p':L\to K$ and that the extension $0 \to \pi_1(S^1) \to \pi_1(L) \to \pi_1(K) \to 1$ is completely determined by a “characteristic class” $\chi \in H^2(\pi_1(K); \mathbb{Z})$. By a theorem of Massey [4, Theorem 2], there is an isomorphism (possibly not the standard isomorphism) $\rho: H^2(\pi_1(K); \mathbb{Z}) \to H^2(K; \mathbb{Z})$ such that $\rho(\chi) = c_1$. The lemma follows.

Let $K = K(\pi_1(B), 1)$ be obtained from $B$ by attaching cells of dimensions $\geq 3$ to kill the higher homotopy of $B$.

**Lemma 1.3.** There is a unique principal $S^1$ bundle $p':L\to K$ whose restriction to $B$ is $p:E\to B$.

**Proof.** Let $f:B\to CP^\infty$ be a classifying map for $p:E\to B$, and let $a:S^3\to B$ be the attaching map for one of the three cells $e^3 \in K - B$. Then $[fa] = f_# [a] \in \pi_2(CP^\infty)$ is the obstruction to extending $f$ over $e^3$. But the commutative diagram

$$\begin{array}{ccc}
\pi_2(B) & \xrightarrow{\partial} & \pi_1(S^1) \\
\downarrow f_# & & \downarrow \cong \\
\pi_2(CP^\infty) & \xrightarrow{\cong} & \pi_1(S^1)
\end{array}$$
shows that $f_{\#}$ is the zero map since $\partial$ is the zero map. Therefore $f$ extends over the 3-skeleton of $K$. Since $CP^\infty$ is a $K(Z, 2)$ this implies that $f$ extends over all of $K$. Therefore there is one bundle over $K$ whose restriction to $B$ is $p : E \to B$.

But also this bundle is unique, for if $f_1, f_2 : K \to K(Z, 2)$ are such that $f_1|B$ is homotopic to $f_2|B$, then $f_1$ is homotopic to $f_2$ since the obstructions to finding such a homotopy lie in $H^3(K, B; \pi_1(CP^\infty))$ and those groups vanish for all $i$.

Now let $B$ be dominated by a finite complex and let $\pi_1(E)$ be abelian. Then $\pi_1(B) \approx \pi_1(K)$ is finitely presented by [8, Lemma 1.3] and the five lemma applied to

$$
\begin{array}{cccccc}
0 & \to & \pi_1(S^1) & \xrightarrow{j_{\#}} & \pi_1(E) & \xrightarrow{p_{\#}} & \pi_1(B) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \simeq & & \\
0 & \to & \pi_1(S^1) & \xrightarrow{j'_{\#}} & \pi_1(L) & \xrightarrow{p'_{\#}} & \pi_1(K) & \to & 0
\end{array}
$$

shows that $\pi_1(L)$ is abelian. Therefore $\pi_1(K)$ is a finitely generated abelian group and may be presented as a direct sum $\mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_s}$ where $r_i = 0$ for $1 \leq i \leq d$, and $r_i > 0$ and $r_i| r_{i+1}$ for $d + 1 \leq i \leq s$ and $\mathbb{Z}_0 = \mathbb{Z}$. Let $c_i \in \mathbb{Z}_{r_i}$ and $ac \approx \pi_1(S^1)$ be generators.

**Lemma 1.4.** Let $b_0, b_1, \cdots, b_s \in \pi_1(L)$ be any elements satisfying $b_0 = j_{\#}(a)$, $p_{\#}(b_i) = c_i$ for $i = 1, \cdots, s$. Then $b_0, b_1, \cdots, b_s$ generate $\pi_1(L)$ and satisfy:

(i) $b_i b_{i+1} b_i^{-1} = 0$.

(ii) $r_i b_i = n_i b_0$ for some integer $n_i$ where $d + 1 \leq i \leq s$.

**Furthermore any relation in $\pi_1(L)$ is a consequence of (i) and (ii).**

**Proof.** This is obvious (cf. [1, Lemma 4.2]).

Define a map of the generators $b_0, b_1, \cdots, b_s$ of $\pi_1(L)$ into $Q$ as follows:

(i) If $r_s = 0$, let $\tau(b_0) = 1$ and $\tau(b_i) = 0$ for $1 \leq i \leq s$.

(ii) If $r_s > 1$, let $\tau(b_0) = 1$, $\tau(b_i) = 0$ for $1 \leq i \leq d$, and $\tau(b_i) = n_i/r_i$ for $d + 1 \leq i \leq s$.

In either case this map of generators respects the relations (i) and (ii) of 1.4 and can therefore be extended to a homomorphism $\tau : \pi_1(L) \to Q$. Note that in case (ii), since $r_i| r_s$ for $d + 1 \leq i \leq s$, $\tau(b_i)$ is a multiple of $1/r_s$. In either case, therefore, the image of $\tau$ is infinite cyclic.

Let $q_K : \hat{K} \to K$ and $q_L : \hat{L} \to L$ be the universal covering spaces of $K$ and $L$ respectively. Let $\pi_1(L)$ act on $\hat{L}$ as the group of covering transformations and define an action of $\pi_1(L)$ on $\hat{K} \times R^1$ by $\alpha(\hat{k}, t) = (p_{\#} \cdot \hat{k}, t + \tau(\alpha))$ where the dot in the first factor indicates the covering transformation action of $\pi_1(K)$ on $\hat{K}$.  

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Proposition 1.5. There is a $\pi_1(L)$ equivariant homeomorphism $h: \tilde{K} \times R^1 \to \tilde{L}$.

It will follow from the construction of $\tilde{h}$, that $\tilde{h}$ maps $\tilde{B} \times R^1$ homeomorphically onto $\tilde{E}$. Thus 1.1 follows from 1.5 by restricting $\tilde{h}$ to $\tilde{B} \times R^1$.

Proof. Let $\pi_1(S^1)$ act on $R^1$ as the group of covering transformations of $e: R^1 \to S^1$ where $e$ is the exponential map. Since $\tau(b_0) = 1$ and $p_\#(b_0) = 0$, $b_0(k, t) = (k, t+1)$ for any $k$ and the diagram

$$
\begin{array}{ccc}
R^1 & \overset{i}{\longrightarrow} & \tilde{K} \times R^1 \\
\downarrow & & \downarrow j'\alpha \\
R^1 & \overset{i}{\longrightarrow} & \tilde{K} \times R^1 \\
\end{array}
$$

commutes for any $\alpha \in \pi_1(S^1)$ where $i(t) = (k_0, t)$ and $k_0 \in \tilde{K}$ is a base point. On the other hand, the diagram

$$
\begin{array}{ccc}
\tilde{K} \times R^1 & \overset{\tilde{\pi}}{\longrightarrow} & \tilde{K} \\
\downarrow \beta & & \downarrow p_\#\beta \\
\tilde{K} \times R^1 & \overset{\tilde{\pi}}{\longrightarrow} & \tilde{K} \\
\end{array}
$$

obviously commutes where $\tilde{\pi}$ is projection on the first factor. Therefore $i$ and $\tilde{\pi}$ induce maps $i: R^1/\pi_1(S^1) \to \tilde{K} \times R^1/\pi_1(L)$ and $\pi: \tilde{K} \times R^1/\pi_1(L) \to \tilde{K}/\pi_1(K)$ respectively such that $\pi i$ is the constant map.

Lemma 1.6. $\pi: \tilde{K} \times R^1/\pi_1(L) \to \tilde{K}/\pi_1(K)$ is a principal $S^1$ bundle with fiber the image of $i$.

Proof. The proof of 1.6 is deferred to the end of this section.

Assuming 1.6, the proof of 1.5 is concluded as follows: By the construction of $\pi: \tilde{K} \times R^1/\pi_1(L) \to \tilde{K}/\pi_1(K)$ and Lemma 1.7 below, the lower end of the homotopy exact sequence of this bundle reduces to

$$
0 \longrightarrow \pi_1(S^1) \overset{j_\#}{\longrightarrow} \pi_1(L) \overset{p'_\#}{\longrightarrow} \pi_1(K) \longrightarrow 0.
$$

By 1.3 therefore, there is a bundle equivalence $h: \tilde{K} \times R^1/\pi_1(L) \to L$. In particular, then, $h$ is a homeomorphism. Since $\pi_1(L)$ acts on $\tilde{K} \times R^1$ as the group of covering transformations of $\tilde{K} \times R^1/\pi_1(L)$, $h$ may be covered by a $\pi_1(L)$ equivariant homeomorphism $\tilde{h}: \tilde{K} \times R^1 \to \tilde{L}$ and the proof of 1.5 is complete.

It is clear from the construction of $\tilde{h}$ that $\tilde{h}(\tilde{B} \times R^1) = \tilde{E}$.

We describe now the context of Lemma 1.7. Let $G$ and $H$ respectively
act freely on $X$ and $Y$ respectively. Let $\tau: G \to H$ be a homomorphism and $f: X \to Y$ be a base point preserving map such that $f(gx) = \tau(g)f(x)$ for any $g \in G$ and $x \in X$. Then $f$ induces a base point preserving map $\tilde{f}: X/G \to Y/H$.

**Lemma 1.7.** If $X$ and $Y$ are simply connected, $f: \pi_1(X/G) \to \pi_1(Y/H)$ corresponds to $\tau: G \to H$ under the canonical identifications of $\pi_1(X/G)$ with $G$ and $\pi_1(Y/H)$ with $H$.

**Proof.** This is obvious.

The only remaining step in the proof of 1.5 is the

**Proof of 1.6.** Let $G \subseteq \pi_1(L)$ be the subgroup generated by $b_0$ and consider the orbit space $\hat{K} \times R^1/G$. Clearly $\hat{K} \times R^1/G = \hat{K} \times S^1$ and the projection map $\hat{K} \times R^1 \to \hat{K} \times R^1/G$ is just $1 \times e: \hat{K} \times R^1 \to \hat{K} \times S^1$ where $e$ is the exponential map.

Since all groups under consideration are abelian (and discrete), there is an induced $\pi_1(L)/G$ action on $\hat{K} \times S^1$ whose orbit space is $\hat{K} \times R^1/\pi_1(L)$. In fact this action can be described explicitly as follows: Since $G$ is contained in the kernel of the composite $\pi_1(L) \xrightarrow{\epsilon} Q \xrightarrow{\epsilon} S^1$, there is an induced homomorphism $\tilde{\tau}: \pi_1(L)/G \to S^1$. Also since $G = \ker \rho'$, there is an induced map $\tilde{\rho}': \pi_1(L)/G \to \pi_1(K)$. The induced action then is given by $\beta(\hat{k}, z) = (\tilde{\rho}'\beta\hat{k}, z\tilde{\tau}(\beta))$ where juxtaposition in the second factor denotes multiplication of complex numbers.

Now define a right action of $S^1$ on $\hat{K} \times S^1$ by $(\hat{k}, z)z_1 = (\hat{k}, zz_1)$. Since this action commutes with the $\pi_1(L)/G$ action described above it induces an $S^1$ action on the orbit space $\hat{K} \times R^1/\pi_1(L)$. It is straightforward to check that the orbit space of the latter action is $\hat{K}/\pi_1(K)$ and that the fiber is the image of $i$. This completes the proof of 1.6.

2. **The proof of the Main Theorem.** Before proving the Main Theorem we recall the definition of the “transfer” homomorphism (cf. [5, p. 420]) and prove two lemmas.

Let $\rho: R \to R$ be a homomorphism of rings with unit and suppose that $R$ is finitely generated and projective as a left $R'$ module. Then any finitely generated projective $R$ module can also be considered as a module over $R'$ and, as such, is finitely generated and projective. Thus $\rho$ induces a homomorphism $\rho^*: \hat{K}_0(R) \to \hat{K}_0(R')$ called the “transfer” homomorphism.

Let $\rho: G \to H$ be a group homomorphism inducing the homomorphism $\rho: Z(G) \to Z(H)$ of group rings.

**Lemma 2.1.** If $\rho$ is a monomorphism with image a subgroup of finite index, $Z(H)$ is finitely generated and projective over $Z(G)$. Hence $\rho^*: \hat{K}_0(Z(H)) \to \hat{K}_0(Z(G))$ is defined.

**Proof.** This is obvious.
Let $C_\bullet$ be a chain complex of $R$ modules. If $C_\bullet$ has the same chain homotopy type as a complex $P_n \to P_{n-1} \to \cdots \to P_0$ of finitely generated projective $R$ modules, we say that $C_\bullet$ admits a Wall invariant and define the invariant as $\sigma(C_\bullet; R) = \sum (-1)^i [P_i] \in K_0(R)$ (cf. [6]).

**Lemma 2.2.** Let $p: R' \to R$ be as above and let $C_\bullet$ be a chain complex of $R$ modules which admits a Wall invariant. Then $C_\bullet$ admits a Wall invariant when considered as a chain complex over $R'$ and $\sigma(C_\bullet; R') = p^* \sigma(C_\bullet; R)$.

**Proof.** This is obvious from the definitions given above.

We now prove the Main Theorem. Because [3, Theorem 1] shows that $E$ is dominated by a finite complex, we restrict our attention to the evaluation of the obstruction.

By [3] or [6], $\sigma(E) = \sigma(C_\bullet(\tilde{E}), Z(\pi_1(E)))$ where $C_\bullet(\tilde{E})$ denotes the singular chain complex of $\tilde{E}$ and the $Z(\pi_1(E))$ action comes from the covering transformation action of $\pi_1(E)$ on $\tilde{E}$. By the first part of 1.1, $\tilde{E}$ is homeomorphic to $\tilde{B} \times R^1$ so $C_\bullet(\tilde{E})$ is isomorphic to $C_\bullet(\tilde{B}) \otimes C_\bullet(R^1)$. Now let $\beta_\circ: C_\bullet(\tilde{B}) \to C_\bullet(\tilde{B})$ and $\eta_\circ: C_\bullet(R^1) \to C_\bullet(R^1)$ be the chain maps induced by the covering transformation $\beta: \tilde{B} \to \tilde{B}$, $\beta \in \pi_1(B)$, and the continuous map $\eta(t) = t + q$ for any $q \in Q$, and define an action of $\pi_1(E)$ on $C_\bullet(\tilde{B}) \otimes C_\bullet(R^1)$ by $\alpha(c' \otimes c^\circ) = (p_\# \alpha)_\bullet c' \otimes \tau(\alpha) c^\circ$ for any $\alpha \in \pi_1(E)$. The latter half of 1.1 then implies that the isomorphism of $C_\bullet(\tilde{E})$ with $C_\bullet(\tilde{B}) \otimes C_\bullet(R^1)$ is an isomorphism over $Z(\pi_1(E))$. Thus

$\sigma(E) = \sigma(C_\bullet(\tilde{B}) \otimes C_\bullet(R^1), Z(\pi_1(E)))$.

On the other hand, since $C_\bullet(\tilde{B})$ is a module over $Z(\pi_1(B))$ and $C_\bullet(R^1)$ is a module over $Z(\tilde{Z})$ where $\tilde{Z}$ is the image of $\tau: \pi_1(E) \to Q$, $C_\bullet(\tilde{B}) \otimes C_\bullet(R^1)$ is also a chain complex over $Z(\pi_1(B)) \otimes Z(\tilde{Z}) \cong Z(\pi_1(B) \times \tilde{Z})$. Since $Z$ is infinite cyclic, $R^1/\tilde{Z} \cong S^1$ and $C_\bullet(R^1)$ considered as a module over $Z(\tilde{Z})$ is nothing more than $C_\bullet(S^1)$ considered as a module over $Z(\pi_1(S^1))$. The Product Theorem for Wall Invariants due to [2] and [7] now shows that $\sigma(C_\bullet(\tilde{B}) \otimes C_\bullet(R^1), Z(\pi_1(B) \times \tilde{Z})) = \chi(S^1) \cdot i_* \sigma(C_\bullet(\tilde{B}), Z(\pi_1(B))) = 0$ where $i_*: K_0(Z(\pi_1(B))) \to K_0(Z(\pi_1(B) \times \tilde{Z}))$ is induced by the inclusion $\pi_1(B) \subset \pi_1(B) \times \tilde{Z}$.

The key observation now is that $p_\# \times \sigma: \pi_1(E) \to \pi_1(B) \times \tilde{Z}$ is a monomorphism with image a subgroup of finite index. By 2.1 therefore $(p_\# \times \sigma)^*: K_0(Z(\pi_1(B) \times \tilde{Z})) \to K_0(Z(\pi_1(E)))$ is defined. The discussion above and 2.2, now show that

$\sigma(C_\bullet(\tilde{B}) \otimes C_\bullet(R^1); Z(\pi_1(E))) = (p_\# \times \tau)^* \sigma(C_\bullet(\tilde{B}) \otimes C_\bullet(R^1); Z(\pi_1(B) \times \tilde{Z})) = 0$.

This completes the proof of the Main Theorem.
Bibliography

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