PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 2, February 1972

THE WALL INVARIANT OF CERTAIN S^1 BUNDLES

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ABSTRACT. Let $p:E\to B$ be a principal S^1 bundle with B dominated by a finite complex. Then it is easy to show that E is also dominated by a finite complex. In this paper we show, under suitable additional hypotheses, that in fact E has the homotopy type of a finite complex. The proof is carried out by computing Wall's finiteness obstruction for E.

Let X be a (possibly infinite) CW complex which is dominated by a finite complex and let $\Lambda=Z(\pi_1(X))$. In [8] Wall defines an obstruction $\sigma(X)\in\tilde{K}_0(\Lambda)$ to finding a finite complex with the same homotopy type as X.

If $F\xrightarrow{j}E\xrightarrow{p}B$ is a fiber space with total space, base, and fiber dominated by finite CW complexes, it is natural to ask whether $\sigma(E)$ can be computed in terms of $\sigma(F)$, $\sigma(B)$, and invariants of the fiber space. Since $\sigma(F)$, $\sigma(E)$, and $\sigma(B)$ are elements of $\tilde{K}_0Z(\pi_1(F))$, $\tilde{K}_0Z(\pi_1(E))$, and $\tilde{K}_0Z(\pi_1(B))$ respectively, and $\cdots\to\pi_1(F)\to\pi_1(E)\to\pi_1(B)\to\cdots$ is exact, the solution of the general problem involves an extension problem. In an effort to obtain a better understanding of the relationship between these problems, we will prove the

MAIN THEOREM. *Let $p:E\to B$ be a principal S^1 bundle such that*
(i) *B is a* CW *complex dominated by a finite complex;*
(ii) *$j_{\#}:\pi_1(S^1)\to\pi_1(E)$ is a monomorphism; and*
(iii) *$\pi_1(E)$ is abelian.*
Then E is dominated by a finite complex and $\sigma(E)=0$.

The universal covering space $\tilde{E}$ of E and the action of $\pi_1(E)$ on $\tilde{E}$ are determined in §1. This information is then used in §2 to prove the theorem.

The problem considered here was first raised by Lal in [3]. Unfortunately, the main theorems of that paper (Theorems 2 and 3) do not hold in the generality Lal claims, but appear to require additional hypotheses. A forthcoming paper by the author [0] contains a counterexample to Lal's Theorems 2 and 3 as well as a modified version of Lal's Theorem 3.

Received by the editors June 9, 1970 and, in revised form, April 23, 1971.
AMS 1970 *subject classifications*. Primary 57C05; Secondary 57C10, 57C50.
[1] Partially supported by the NSF under grant number GP12837.

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1. **$\tilde{E}$ and the $\pi_1(E)$ action on $\tilde{E}$.** It is the object of this section to prove the

THEOREM 1.1. *Let $p:E\to B$ be a principal S^1 bundle satisfying the hypothesis of the Main Theorem. Then $\tilde{E}$ is homeomorphic to $\tilde{B}\times R^1$. Furthermore there is a homomorphism $\tau:\pi_1(E)\to Q$ with image an infinite cyclic subgroup of Q and a choice of homeomorphism $\tilde{h}:\tilde{B}\times R^1\to\tilde{E}$ such that $\tilde{h}$ is equivariant with respect to the $\pi_1(E)$ action on $\tilde{B}\times R^1$ defined by $\alpha(\tilde{b}, t)=(p_{\#}\alpha\tilde{b}, t+\tau(\alpha))$ for any $\alpha\in\pi_1(E)$ and the covering transformation action of $\pi_1(E)$ on $\tilde{E}$.*

The Q above denotes the additive group of rational numbers and the juxtaposition $p_{\#}\alpha\tilde{b}$ denotes the covering transformation action of $\pi_1(B)$ on $\tilde{B}$.

The basic idea of the proof of 1.1 is to apply the

LEMMA 1.2. *Let $p':L\to K$ be a principal S^1 bundle with base space a $K(\pi_1(K), 1)$. The isomorphism class of the extension*

$$0\longrightarrow \pi_1(S^1)\xrightarrow{j'_{\#}}\pi_1(L)\xrightarrow{p'_{\#}}\pi_1(K)\longrightarrow 1$$

is a complete invariant of the bundle.

PROOF. It is well known that the Chern class $C_1\in H^2(K; Z)$ is a complete invariant of the bundle $p':L\to K$ and that the extension $0\to\pi_1(S^1)\to\pi_1(L)\to\pi_1(K)\to 1$ is completely determined by a "characteristic class" $\chi\in H^2(\pi_1(K); Z)$. By a theorem of Massey [**4**, Theorem 2], there is an isomorphism (possibly not the standard isomorphism) $\rho:H^2(\pi_1(K);Z)\to H^2(K; Z)$ such that $\rho(\chi)=C_1$. The lemma follows.

Let $K=K(\pi_1(B), 1)$ be obtained from B by attaching cells of dimensions $\geqq 3$ to kill the higher homotopy of B.

LEMMA 1.3. *There is a unique principal S^1 bundle $p':L\to K$ whose restriction to B is $p:E\to B$.*

PROOF. Let $f:B\to CP^\infty$ be a classifying map for $p:E\to B$, and let $a:S^2\to B$ be the attaching map for one of the three cells $e^3\in K-B$. Then $[fa]=f_{\#}[a]\in\pi_2(CP^\infty)$ is the obstruction to extending f over e^3. But the commutative diagram

$$\begin{array}{ccc} \pi_2(B) & \xrightarrow{\partial} & \pi_1(S^1) \\ \downarrow{\scriptstyle f_{\#}} & & \downarrow{\scriptstyle \approx} \\ \pi_2(CP^\infty) & \xrightarrow{\approx} & \pi_1(S^1) \end{array}$$

shows that $f_\#$ is the zero map since ∂ is the zero map. Therefore f extends over the 3-skeleton of K. Since CP^∞ is a $K(Z, 2)$ this implies that f extends over all of K. Therefore there is one bundle over K whose restriction to B is $p: E \to B$.

But also this bundle is unique, for if $f_1, f_2: K \to K(Z, 2)$ are such that $f_1|B$ is homotopic to $f_2|B$, then f_1 is homotopic to f_2 since the obstructions to finding such a homotopy lie in $H^i(K, B; \pi_i(CP^\infty))$ and those groups vanish for all i.

Now let B be dominated by a finite complex and let $\pi_1(E)$ be abelian. Then $\pi_1(B) \approx \pi_1(K)$ is finitely presented by [**8**, Lemma 1.3] and the five lemma applied to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{j_\#} & \pi_1(E) & \xrightarrow{p_\#} & \pi_1(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle \approx} & & \\ 0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{j'_\#} & \pi_1(L) & \xrightarrow{p'_\#} & \pi_1(K) & \longrightarrow & 0 \end{array}$$

shows that $\pi_1(L)$ is abelian. Therefore $\pi_1(K)$ is a finitely generated abelian group and may be presented as a direct sum $Z_{r_1} \oplus \cdots \oplus Z_{r_s}$ where $r_i = 0$ for $1 \leqq i \leqq d$, and $r_i > 0$ and $r_i \mid r_{i+1}$ for $d+1 \leqq i \leqq s$ and $Z_0 = Z$. Let $c_i \in Z_{r_i}$ and $a \in Z \approx \pi_1(S^1)$ be generators.

LEMMA 1.4. *Let $b_0, b_1, \cdots, b_s \in \pi_1(L)$ be any elements satisfying $b_0 = j'_\#(a)$, $p'_\#(b_i) = c_i$ for $i = 1, \cdots, s$. Then $b_0, b_1, \cdots, b_s$ generate $\pi_1(L)$ and satisfy*:

(i) $b_i b_j b_i^{-1} b_j^{-1} = 0$.

(ii) $r_i b_i = n_i b_0$ *for some integer n_i where $d+1 \leqq i \leqq s$.*

Furthermore any relation in $\pi_1(L)$ is a consequence of (i) *and* (ii).

PROOF. This is obvious (cf. [**1**, Lemma 4.2]).

Define a map of the generators $b_0, b_1, \cdots, b_s$ of $\pi_1(L)$ into Q as follows:

(i) If $r_s = 0$, let $\tau(b_0) = 1$ and $\tau(b_i) = 0$ for $1 \leqq i \leqq s$.

(ii) If $r_s > 1$, let $\tau(b_0) = 1$, $\tau(b_i) = 0$ for $1 \leqq i \leqq d$, and $\tau(b_i) = n_i / r_i$ for $d+1 \leqq i \leqq s$.

In either case this map of generators respects the relations (i) and (ii) of 1.4 and can therefore be extended to a homomorphism $\tau: \pi_1(L) \to Q$. Note that in case (ii), since $r_i | r_s$ for $d+1 \leqq i \leqq s$, $\tau(b_i)$ is a multiple of $1/r_s$. In either case, therefore, the image of τ is infinite cyclic.

Let $q_K: \tilde{K} \to K$ and $q_L: \tilde{L} \to L$ be the universal covering spaces of K and L respectively. Let $\pi_1(L)$ act on $\tilde{L}$ as the group of covering transformations and define an action of $\pi_1(L)$ on $\tilde{K} \times R^1$ by $\alpha(\tilde{k}, t) = (p_\# \alpha \cdot \tilde{k}, t + \tau(\alpha))$ where the dot in the first factor indicates the covering transformation action of $\pi_1(K)$ on $\tilde{K}$.

PROPOSITION 1.5. *There is a $\pi_1(L)$ equivariant homeomorphism $\tilde{h}$: $\tilde{K}\times R^1\to\tilde{L}$.*

It will follow from the construction of $\tilde{h}$, that $\tilde{h}$ maps $\tilde{B}\times R^1$ homeomorphically onto $\tilde{E}$. Thus 1.1 follows from 1.5 by restricting $\tilde{h}$ to $\tilde{B}\times R^1$.

PROOF. Let $\pi_1(S^1)$ act on R^1 as the group of covering transformations of $e: R^1\to S^1$ where e is the exponential map. Since $\tau(b_0)=1$ and $p_\#(b_0)=0$, $b_0(\tilde{k}, t)=(\tilde{k}, t+1)$ for any $\tilde{k}$ and the diagram

$$\begin{array}{ccc} R^1 & \xrightarrow{\tilde{\imath}} & \tilde{K}\times R^1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle j'_\#\alpha} \\ R^1 & \xrightarrow{\tilde{\imath}} & \tilde{K}\times R^1 \end{array}$$

commutes for any $\alpha\in\pi_1(S^1)$ where $\tilde{\imath}(t)=(\tilde{k}_0, t)$ and $\tilde{k}_0\in\tilde{K}$ is a base point. On the other hand, the diagram

$$\begin{array}{ccc} \tilde{K}\times R^1 & \xrightarrow{\tilde{\pi}} & \tilde{K} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle p'_\#\beta} \\ \tilde{K}\times R^1 & \xrightarrow{\tilde{\pi}} & \tilde{K} \end{array}$$

obviously commutes where $\tilde{\pi}$ is projection on the first factor. Therefore $\tilde{\imath}$ and $\tilde{\pi}$ induce maps $i: R^1/\pi_1(S^1)\to\tilde{K}\times R^1/\pi_1(L)$ and $\pi:\tilde{K}\times R^1/\pi_1(L)\to\tilde{K}/\pi_1(K)$ respectively such that πi is the constant map.

LEMMA 1.6. *$\pi:\tilde{K}\times R^1/\pi_1(L)\to\tilde{K}/\pi_1(K)$ is a principal S^1 bundle with fiber the image of i.*

PROOF. The proof of 1.6 is deferred to the end of this section.

Assuming 1.6, the proof of 1.5 is concluded as follows: By the construction of $\pi:\tilde{K}\times R^1/\pi_1(L)\to\tilde{K}/\pi_1(K)$ and Lemma 1.7 below, the lower end of the homotopy exact sequence of this bundle reduces to

$$0\longrightarrow\pi_1(S^1)\xrightarrow{j'_\#}\pi_1(L)\xrightarrow{p'_\#}\pi_1(K)\longrightarrow 0.$$

By 1.3 therefore, there is a bundle equivalence $h:\tilde{K}\times R^1/\pi_1(L)\to L$. In particular, then, h is a homeomorphism. Since $\pi_1(L)$ acts on $\tilde{K}\times R^1$ as the group of covering transformations of $\tilde{K}\times R^1/\pi_1(L)$, h may be covered by a $\pi_1(L)$ equivariant homeomorphism $\tilde{h}:\tilde{K}\times R^1\to\tilde{L}$ and the proof of 1.5 is complete.

It is clear from the construction of $\tilde{h}$ that $\tilde{h}(\tilde{B}\times R^1)=\tilde{E}$.

We describe now the context of Lemma 1.7. Let G and H respectively act freely on X and Y respectively. Let $\tau: G \to H$ be a homomorphism and $f: X \to Y$ be a base point preserving map such that $f(gx) = \tau(g) f(x)$ for any $g \in G$ and $x \in X$. Then f induces a base point preserving map $\bar{f}: X/G \to Y/H$.

LEMMA 1.7. *If X and Y are simply connected, $\bar{f}_{\#}: \pi_1(X/G) \to \pi_1(Y/H)$ corresponds to $\tau: G \to H$ under the canonical identifications of $\pi_1(X/G)$ with G and $\pi_1(Y/H)$ with H.*

PROOF. This is obvious.

The only remaining step in the proof of 1.5 is the

PROOF OF 1.6. Let $G \subset \pi_1(L)$ be the subgroup generated by b_0 and consider the orbit space $\tilde{K} \times R^1/G$. Clearly $\tilde{K} \times R^1/G = \tilde{K} \times S^1$ and the projection map $\tilde{K} \times R^1 \to \tilde{K} \times R^1/G$ is just $1 \times e: \tilde{K} \times R^1 \to \tilde{K} \times S^1$ where e is the exponential map.

Since all groups under consideration are abelian (and discrete), there is an induced $\pi_1(L)/G$ action on $\tilde{K} \times S^1$ whose orbit space is $\tilde{K} \times R^1/\pi_1(L)$. In fact this action can be described explicitly as follows: Since G is contained in the kernel of the composite $\pi_1(L) \xrightarrow{\tau} Q \xrightarrow{e} S^1$, there is an induced homomorphism $\bar{\tau}: \pi_1(L)/G \to S^1$. Also since $G = \ker p'_{\#}$, there is an induced map $\bar{p}'_{\#}: \pi_1(L)/G \to \pi_1(K)$. The induced action then is given by $\beta(\tilde{k}, z) = (\bar{p}'_{\#}\beta\tilde{k}, z\bar{\tau}(\beta))$ where juxtaposition in the second factor denotes multiplication of complex numbers.

Now define a right action of S^1 on $\tilde{K} \times S^1$ by $(\tilde{k}, z)z_1 = (\tilde{k}, zz_1)$. Since this action commutes with the $\pi_1(L)/G$ action described above it induces an S^1 action on the orbit space $\tilde{K} \times R^1/\pi_1(L)$. It is straightforward to check that the orbit space of the latter action is $\tilde{K}/\pi_1(K)$ and that the fiber is the image of i. This completes the proof of 1.6.

2. **The proof of the Main Theorem.** Before proving the Main Theorem we recall the definition of the "transfer" homomorphism (cf. [**5**, p. 420]) and prove two lemmas.

Let $\rho: R' \to R$ be a homomorphism of rings with unit and suppose that R is finitely generated and projective as a left R' module. Then any finitely generated projective R module can also be considered as a module over R' and, as such, is finitely generated and projective. Thus ρ induces a homomorphism $\rho^*: \tilde{K}_0(R) \to \tilde{K}_0(R')$ called the "transfer" homomorphism.

Let $\bar{\rho}: G \to H$ be a group homomorphism inducing the homomorphism $\rho: Z(G) \to Z(H)$ of group rings.

LEMMA 2.1. *If $\bar{\rho}$ is a monomorphism with image a subgroup of finite index, $Z(H)$ is finitely generated and projective over $Z(G)$. Hence ρ^*: $\tilde{K}_0(Z(H)) \to \tilde{K}_0(Z(G))$ is defined.*

PROOF. This is obvious.

Let C_* be a chain complex of R modules. If C_* has the same chain homotopy type as a complex $P_n \to P_{n-1} \to \cdots \to P_0$ of finitely generated projective R modules, we say that C_* *admits a Wall invariant* and define the invariant as $\sigma(C_*; R) = \sum (-1)^i [P_i] \in \tilde{K}_0(R)$ (cf. [6]).

LEMMA 2.2. *Let $\rho\colon R' \to R$ be as above and let C_* be a chain complex of R modules which admits a Wall invariant. Then C_* admits a Wall invariant when considered as a chain complex over R' and $\sigma(C_*; R') = \rho^*\sigma(C_*; R)$.*

PROOF. This is obvious from the definitions given above.

We now prove the Main Theorem. Because [3, Theorem 1] shows that E is dominated by a finite complex, we restrict our attention to the evaluation of the obstruction.

By [3] or [6], $\sigma(E) = \sigma(C_*(\tilde{E}), Z(\pi_1(E)))$ where $C_*(\tilde{E})$ denotes the singular chain complex of $\tilde{E}$ and the $Z(\pi_1(E))$ action comes from the covering transformation action of $\pi_1(E)$ on $\tilde{E}$. By the first part of 1.1, $\tilde{E}$ is homeomorphic to $\tilde{B} \times R^1$ so $C_*(\tilde{E})$ is isomorphic to $C_*(\tilde{B}) \otimes C_*(R^1)$. Now let $\beta_*\colon C_*(\tilde{B}) \to C_*(\tilde{B})$ and $\bar{q}_*\colon C_*(R^1) \to C_*(R^1)$ be the chain maps induced by the covering transformation $\beta\colon \tilde{B} \to \tilde{B}$, $\beta \in \pi_1(B)$, and the continuous map $\bar{q}(t) = t + q$ for any $q \in Q$, and define an action of $\pi_1(E)$ on $C_*(\tilde{B}) \otimes C_*(R^1)$ by $\alpha(c' \otimes c'') = (p_{\#}\alpha)_* c' \otimes \tau(\alpha)_* c''$ for any $\alpha \in \pi_1(E)$. The latter half of 1.1 then implies that the isomorphism of $C_*(\tilde{E})$ with $C_*(\tilde{B}) \otimes C_*(R^1)$ is an isomorphism over $Z(\pi_1(E))$. Thus

$$\sigma(E) = \sigma(C_*(\tilde{B}) \otimes C_*(R^1), Z(\pi_1(E))).$$

On the other hand, since $C_*(\tilde{B})$ is a module over $Z(\pi_1(B))$ and $C_*(R^1)$ is a module over $Z(\boldsymbol{Z})$ where $\boldsymbol{Z}$ is the image of $\tau\colon \pi_1(E) \to Q$, $C_*(\tilde{B}) \otimes C_*(R^1)$ is also a chain complex over $Z(\pi_1(B)) \otimes Z(\boldsymbol{Z}) \approx Z(\pi_1(B) \times \boldsymbol{Z})$. Since $\boldsymbol{Z}$ is infinite cyclic, $R^1/\boldsymbol{Z} \approx S^1$ and $C_*(R^1)$ considered as a module over $Z(\boldsymbol{Z})$ is nothing more than $C_*(\tilde{S}^1)$ considered as a module over $Z(\pi_1(S^1))$. The Product Theorem for Wall Invariants due to [2] and [7] now shows that $\sigma(C_*(\tilde{B}) \otimes C_*(R^1), Z(\pi_1(B) \times \boldsymbol{Z})) = \chi(S^1) \cdot i_* \sigma(C_*(\tilde{B}), Z(\pi_1(B))) = 0$ where $i_*\colon \tilde{K}_0(Z(\pi_1(B))) \to \tilde{K}_0(Z(\pi_1(B) \times \boldsymbol{Z}))$ is induced by the inclusion $\pi_1(B) \subset \pi_1(B) \times \boldsymbol{Z}$.

The key observation now is that $p_{\#} \times \sigma\colon \pi_1(E) \to \pi_1(B) \times \boldsymbol{Z}$ is a monomorphism with image a subgroup of finite index. By 2.1 therefore $(p_{\#} \times \sigma)^*\colon \tilde{K}_0(Z(\pi_1(B) \times \boldsymbol{Z})) \to \tilde{K}_0(Z(\pi_1(E)))$ is defined. The discussion above and 2.2, now show that

$$\begin{aligned} &\sigma(C_*(\tilde{B}) \otimes C_*(R^1); Z(\pi_1(E))) \\ &\qquad = (p_{\#} \times \tau)^* \sigma(C_*(\tilde{B}) \otimes C_*(R^1); Z(\pi_1(B) \times \boldsymbol{Z})) = 0. \end{aligned}$$

This completes the proof of the Main Theorem.

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