ALGEBRAIC ALGEBRAS WITH INVOLUTION

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ABSTRACT. The following theorem is proved: Let $R$ be an algebra with involution over an uncountable field $F$. Then if the symmetric elements of $R$ are algebraic, $R$ is algebraic.

In this paper we consider the following question:

"Let $R$ be an algebra with involution over a field $F$, and assume that the symmetric elements $S$ of $R$ are algebraic over $F$. Is $R$ algebraic over $F$?"

Previous results related to this question have been obtained by restricting the kind of algebraic relationships satisfied by the symmetric elements. For example, it was shown by Baxter and Martindale [1] for fields of characteristic not 2, and later by the author [5] for arbitrary fields, that if the symmetric elements are algebraic of bounded degree (or more generally, satisfy a polynomial identity), then $R$ must be algebraic. Another such result concerns rings whose symmetric elements are periodic (that is, for each $s \in S$, there is some integer $n(s) > 1$ such that $s^{n(s)} = s$). In this case, the author has shown [6], [7] that $R$ must be algebraic; in fact $R$ satisfies a polynomial identity. When $R$ is a division ring, much more can be said: I. N. Herstein and the author [2] have shown that $R$ must actually be commutative. Finally, it has been shown by Osborn [8] that if $S$ is nil and $F$ is uncountable, then $R$ is nil. This answers for uncountable fields a question of McCrimmon [4, p. 391]:

"If $S$ is nil, is $R$ nil?"

An affirmative answer to this question in general would follow from an affirmative answer to the first question. For, as has been observed by both McCrimmon [4, p. 390] and Osborn [8, p. 306], if $S$ is nil then $R$ must be a radical ring. But if $R$ is algebraic, every element of the radical is nil; thus $R$ would be nil.

The result presented here differs from those described above in that no additional restrictions are imposed on the symmetric elements. We prove:

THEOREM. Let $R$ be an algebra with involution over an uncountable field $F$. Then if the symmetric elements of $R$ are algebraic, $R$ is algebraic.

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If $R$ is a ring, an involution on $R$ is simply an anti-automorphism of period 2. By an algebra with involution, we mean that $R$ has an involution $*$ as a ring, and that the field $F$ has an automorphism $\alpha \mapsto \bar{\alpha}$ of period 2 such that $(\bar{\alpha}r)\bar{\alpha} = \bar{\alpha}r*$, for all $\alpha \in F$, $r \in R$. $S = \{x \in R | x^* = x\}$ will denote the symmetric elements of $R$.

**Lemma 1.** Let $R$ be an algebra with unit over a field $F$, and say $x \in R$ with $x^2 = rx + s$, $r, s \in R$. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the $2 \times 2$ matrix. Then if $A$ is algebraic over $F$, $x$ is algebraic over $F$.

**Proof.** We first notice that $A^2 = rA + sI$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus

$$A^3 = (rA + sI)A = r(rA + sI) + sA = (r^2 + s)A + rsI = r_1A + s_1I.$$ 

Similarly, $A^n = r_{n-2}A + s_{n-2}I$, where $r_{n-2}, s_{n-2} \in R$ for all $n > 2$. Since $x$ satisfies $x^2 = rx + s$, by the same procedure as for $A$ we find that also $x^n = r_{n-2}x + s_{n-2}$, for all $n > 2$. Now if $A$ is algebraic over $F$, there exists some polynomial $p(\lambda) \in F[\lambda]$ such that $p(A) = 0$. We claim that $p(x) = 0$.

For, if $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$, $\alpha_i \in F$, then

$$0 = p(A)$$

$$= (r_{n-2}A + s_{n-2}I) + \alpha_{n-1}(r_{n-3}A + s_{n-3}I)$$

$$+ \cdots + \alpha_2(rA + sI) + \alpha_1A + \alpha_0I$$

$$= (r_{n-2} + \alpha_{n-1}r_{n-3} + \cdots + \alpha_2r + \alpha_1)A$$

$$+ (s_{n-2} + \alpha_{n-1}s_{n-3} + \cdots + \alpha_2s + \alpha_0)I$$

$$= tA + t'I,$$ 

where $t, t' \in R$. But

$$tA + t'I = \begin{pmatrix} 0 & t \\ ts & tr \end{pmatrix} + \begin{pmatrix} t' & 0 \\ 0 & t' \end{pmatrix} = \begin{pmatrix} t' & t \\ ts & t' + tr \end{pmatrix},$$

so $tA + t'I = 0$ implies $t = 0$ and $t' = 0$. Since $x^i = r_{i-2}x + s_{i-2}$, $i > 2$, $p(x) = tx + t' = 0$ and thus $x$ is algebraic.

Recall that if $R$ is any algebra with unit, we may consider $R$ as an algebra of linear transformations by letting $R$ act on itself by right multiplication. Thus a characteristic root (or vector) of an element $r \in R$ will mean a characteristic root (or vector) of $r$ considered as a linear transformation acting by right multiplication. For any $r \in R$, we also define the spectrum $\sigma(r) = \{x \in F \text{ such that } r - \alpha 1 \text{ has no inverse in } R\}$. The resolvent $\rho(r)$ is the complement of $\sigma(r)$ in $F$.

**Lemma 2.** Let $R$ be an algebra with involution over any field $F$ such that $S$ is algebraic. Assume that $R$ has a unit element, and that $F$ is fixed element-wise by $\bar{\cdot}$. Choose $x \in R$, and let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $r = x + x^*$ and $s = -x^*x$. 

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Consider $r$ acting by right multiplication on $R$, and $A$ acting by right multiplication on the $2 \times 2$ matrices over $R$. Then for any $\alpha \in \rho(r)$, the resolvent of $r$, with $\alpha \neq 0$, either $\alpha$ is a characteristic root of $A$ or $\alpha \in \rho(A)$, the resolvent of $A$.

**Proof.** Let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $A = a + y$. Let $\alpha \in F$, $\alpha \neq 0$. Assume that $a - \alpha I$ is invertible in $R_2$. Then there is some matrix $\begin{pmatrix} b & c \\ d & e \end{pmatrix}$, $b, c, d, e \in R$, such that $(a - \alpha I)(b \ d) = I$. Since $a - \alpha I = \begin{pmatrix} \alpha^{-1} & -1 \\ 0 & 0 \end{pmatrix}$, this gives

$$
\begin{pmatrix} -\alpha & 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} b & c \\ d & e \end{pmatrix} = \begin{pmatrix} -\alpha b + d & -\alpha c + e \\ (r - \alpha)d & (r - \alpha)e \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

In particular, $(r - \alpha)e = 1$ and so $(r - \alpha)^{-1}$ exists in $R$. Also $d = 0$, $b = -1/\alpha$, and $c = (r - \alpha)^{-1}$, and thus

$$(a - \alpha I)^{-1} = \begin{pmatrix} -1 & 1 \\ \alpha & \alpha \end{pmatrix} (r - \alpha)^{-1} \begin{pmatrix} 0 & 0 \\ -\frac{s}{\alpha} & \frac{s}{\alpha} \end{pmatrix}.$$

Certainly if $\alpha \neq 0$ and $(r - \alpha)^{-1}$ exists in $R$, we have that $(a - \alpha I)^{-1}$ exists in $R_2$ by the expression for $(a - \alpha I)^{-1}$.

To summarize: if $\alpha \neq 0$, $a - \alpha I$ is invertible if and only if $r - \alpha$ is invertible. This means that $\rho(a) \subseteq \rho(r)$; in fact if $0 \notin \rho(r)$, $\rho(a) = \rho(r)$, and if $0 \in \rho(r)$, $\rho(r) = \rho(a) \cup \{0\}$.

Choose $\alpha \in \rho(r)$, $\alpha \neq 0$. Consider $A - \alpha I = (a - \alpha I) + y$. Multiplying on the right by $(a - \alpha I)^{-1}$ gives $I + y(a - \alpha I)^{-1} = I + y'$, where $y' = y(a - \alpha I)^{-1}$. We claim that $y'$ is algebraic. For,

$$
y' = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \begin{pmatrix} -\frac{s}{\alpha} & \frac{s}{\alpha} \\ 0 & (r - \alpha)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{s}{\alpha} & \frac{s}{\alpha} \end{pmatrix} (r - \alpha)^{-1}.$$

Now $(s/\alpha)(r - \alpha)^{-1} = -(x^*x/\alpha)(r - \alpha)^{-1}$. Since $r - \alpha \in S$, $(r - \alpha)^{-1} \in S$ and thus $x(-(r - \alpha)^{-1}/\alpha)x^* \in S$. But then $x(-(r - \alpha)^{-1}/\alpha)x^*$ is algebraic, and so $x^*x(-(r - \alpha)^{-1}/\alpha) = (s/\alpha)(r - \alpha)^{-1}$ is algebraic. Say $t((s/\alpha)(r - \alpha)^{-1}) = 0$, some polynomial $t(\lambda)$. We may assume that $t(\lambda)$ has no constant term (multiply by $\lambda$ if necessary). Thus $t(y') \in (x^*)R_2$, so $t(y')^2 = 0$, and $y'$ is algebraic.

For an algebraic element, the spectrum coincides with the characteristic roots of the linear transformation [3, p. 246]. Hence either $-1 \in \rho(y')$ or $-1$ is a characteristic root of $y'$.

If $-1 \in \rho(y')$, then $(I + y')^{-1}$ exists, and so

$$(a + y) - \alpha I)^{-1} = (a - \alpha I)^{-1}(I + y')^{-1}.$$
and $\alpha \in \rho(A)$. But if $-1$ is a characteristic root of $y'$, then there is an $x \neq 0$, $x \in \mathbb{R}_2$, so that $x(I+y')=0$. Then $x(a+y-aI) = x(I+y')(a-aI)=0$, and $x$ is a characteristic root of $A$.

**Proof of the Theorem.** Choose $x \in \mathbb{R}$. Then $x^2-(x+x^*)x+x^*x=0$; letting $r = x + x^*$ and $s = -x^*x$, we have $x^2 = rx + s$. Thus by Lemma 1, it is enough to show that $A = \begin{pmatrix} 0 & 1 \\ s & r \end{pmatrix}$ is algebraic.

First note that we may assume that $F$ is left elementwise fixed by the automorphism $-$. For if not, let $F_0$ be the subfield of $F$ fixed elementwise by $-$. $R$ is certainly an algebra over $F_0$, $F_0$ is uncountable, and $F$ is algebraic over $F_0$ (as $-^{-1}$ has period 2). Thus if $s \in S$ is algebraic over $F$, $s$ is algebraic over $F_0$. Thus $R$ satisfies the hypotheses as an algebra over $F_0$. But if $R$ is algebraic over $F_0$, $R$ is certainly algebraic over $F$.

We may also assume that $R$ contains a unit element. For if not, consider the algebra $R_1 = \{(r, \alpha) | r \in R, \alpha \in F\}$, where addition is defined componentwise and multiplication is given by $(r, \alpha) \cdot (t, \beta) = (rt + \alpha t + \beta r, \alpha \beta)$. $R_1$ has an involution, given by $(r, \alpha)^* = (r^*, \alpha)$, and is an algebra over $F$ by $\alpha(r, \beta) = (\alpha r, \alpha \beta)$. Now the symmetric elements of $R_1$ are algebraic: let $(s, \alpha)$ be a symmetric element. Since $s \in S$, $s$ is algebraic over $F$, say $p(s) = 0$, where $p(\lambda) \in F[\lambda]$. Then $(s, \alpha)$ satisfies the polynomial $p(\lambda - \alpha)$, so is algebraic. Certainly if $R_1$ is algebraic, $R$ is algebraic.

Finally, we may assume that $R$ is finitely generated over $F$—if not, replace $R$ by $R' = F[1, x, x^*]$. This means that the dimension of $R$ over $F$ is countable.

We apply Lemma 2 to see that for any $\alpha \in \rho(r)$, either $\alpha$ is a characteristic root of $A$ or $\alpha \in \rho(A)$. But $\rho(r)$ is uncountable, since the spectrum of $r$ consists of the roots of its minimal polynomial [3, p. 20], and so either $\rho(A)$ is uncountable or the set of distinct characteristic roots of $A$ is uncountable. The latter is impossible, for then $R_2$, the $2 \times 2$ matrices, would contain an uncountable number of characteristic vectors, which are linearly independent. This contradicts the dimension of $R_2$ over $F$ being countable.

Thus it must be that $\rho(A)$ is uncountable, and so $A$ is algebraic [3, p. 20].

**Added in Proof.** Kevin McCrimmon now has a more direct proof of the theorem.

**References**


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