POLYÁ'S PROPERTY W AND FACTORIZATION—
A SHORT PROOF

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ABSTRACT. For an nth order linear differential expression, the
equivalence of Pólya's Property W and factorization into first order
expressions is proven directly and briefly.

The usual proof that a linear differential expression with Pólya's
Property W admits a factorization into first order expressions employs
Jacobi's formula for the minors of the adjugate matrix [1]. (For Jacobi's
theorem, see [2].) A simple, direct proof is given here based on the
following two elementary lemmas.

**Lemma 1.** If $I$ is an open interval, $f \in C^1(I)$, $f$ is nonzero on $I$, and

$$J(y) = f(d/dx)(y^{-1})$$

then $J(f) = 0$ and $J$ is a first order linear differential expression with leading
coefficient equal to one.

**Proof.** The lemma is verified by an easy computation.

**Notation.** If $\{h_k\}_{k=1}^m \subset C^m(I)$, we let $W(h_1, \cdots, h_m)$ denote the
Wronskian determinant of this set of functions.

**Lemma 2.** If $\{h_k\}_{k=1}^m \subset C^m(I)$, $W(h_1, \cdots, h_m)(x) \neq 0$, $\forall x \in I$, and

$$K(y) = W(h_1, h_2, \cdots, h_m, y)[W(h_1, h_2, \cdots, h_m)]^{-1}$$

then $K$ is the unique mth order linear differential expression with leading
coefficient equal to one for which $\{h_k\}_{k=1}^m$ is a fundamental set.

**Proof.** Expansion of the determinant in terms of $y^{(k)}$, $k = 0, \cdots, m$,
and the corresponding cofactors verifies the form of $K(y)$. If $K_1$ were a
second such expression, then $K(y) - K_1(y)$ would be of order $(m-1)$ with
m linearly independent solutions.

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expression.
Theorem. Let \( L(y) \) be a linear differential expression of order \( n \) with continuous coefficients and with leading coefficient equal to one. Let \( \{h_k\}_{k=1}^n \) be a fundamental set for \( L \) such that \( W_m(x) = W(h_1, \ldots, h_m)(x) \neq 0 \), \( \forall x \in I \), \( m = 1, \ldots, n-1 \). Then

\[
L(y) = \frac{W_n}{W_{n-1}} \frac{d}{dx} \frac{W_{n-2}}{W_{n-2}} \frac{d}{dx} \cdots \frac{d}{dx} \frac{W_2}{W_1} \frac{d}{dx} \frac{W_1}{W_1} \frac{d}{dx} \frac{y}{W_1}.
\]

Proof. Set \( W_0 = 1 \). Set

\[
L_k(y) = \frac{W_k}{W_{k-1}} \frac{d}{dx} \left( \frac{W_{k-1}}{W_k} y \right), \quad k = 1, \ldots, n.
\]

Then \( L_1(h_1) = 0 \), and \( L_1 \) has leading coefficient equal to one. Inductively, we assume \( L_k(L_{k-1}(\cdots L_2(L_1(y))\cdots)) = (\prod_{i=1}^k L_i)(y) \) maps \( h_m \) to zero for \( m = 1, \ldots, k \) and has leading coefficient equal to one. Then

\[
(\prod_{i=1}^k L_i)(h_{k+1}) = W_{k+1}^{-1} W_k
\]

by Lemma 2. So \( L_{k+1}(\prod_{i=1}^k L_i(h_{k+1})) = 0 \) by Lemma 1. The composition of two expressions with leading coefficient equal to one is again of that form. Thus \( \prod_{i=1}^n L_i \) has \( \{h_k\}_{k=1}^n \) as a fundamental set. Hence, by Lemma 2, \( \prod_{i=1}^n L_i = L \).

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References


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